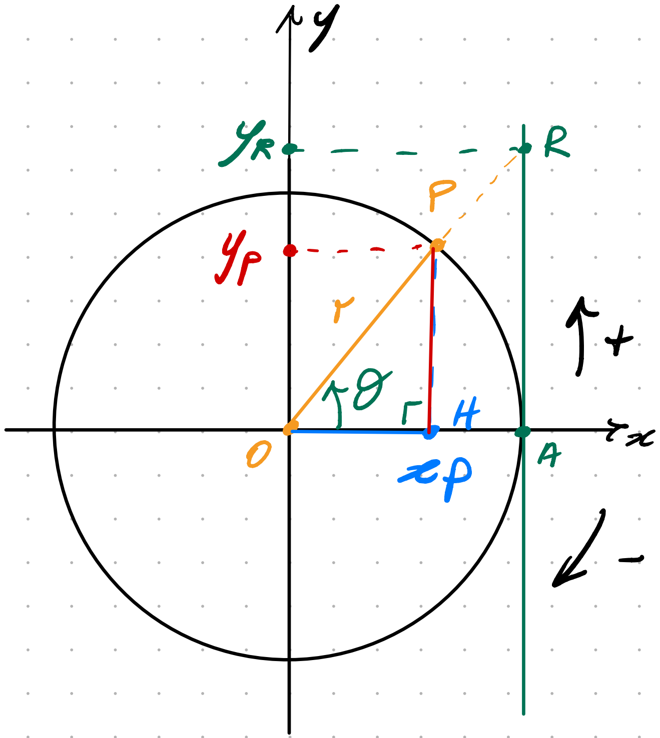
LEZIONE 6 EQUAZIONI E DISEQUAZIONI TRIGONOMETRICHE

RECAP



$$\operatorname{sen}\theta = \frac{\overline{PH}}{r} = \frac{y_P}{r}$$

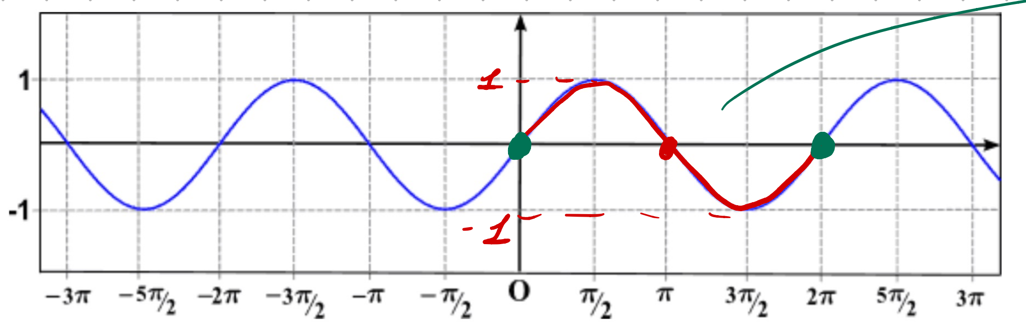
$$\cos\theta = \frac{\overline{OH}}{r} = \frac{x_P}{r}$$

$$\tan\theta = \frac{\overline{PH}}{\overline{OH}} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{\overline{RA}}{r} = \frac{y_R}{r}$$

Con $r = 1$ $\Rightarrow$ $\operatorname{sen}\theta = y_P,\ \cos\theta = x_P,\ \tan\theta = y_R$

↳ $x^2 + y^2 = 1$ $\Rightarrow$ $\sin^2\theta + \cos^2\theta = 1$

GRAFICI FUNZIONI

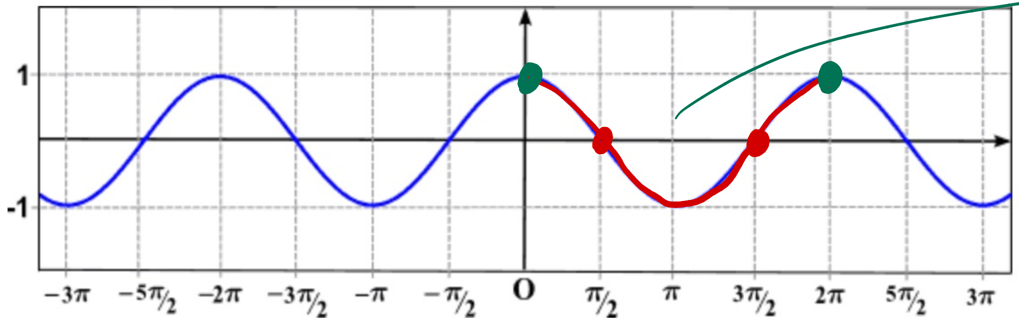


PERIODO 2π

$y = \sin(x)$

$\sin(x) : \mathbb{R} \rightarrow [-1, 1]$

$\sin(x) = 0 \quad \forall\, x = k\pi \,, \quad k \in \mathbb{Z}$

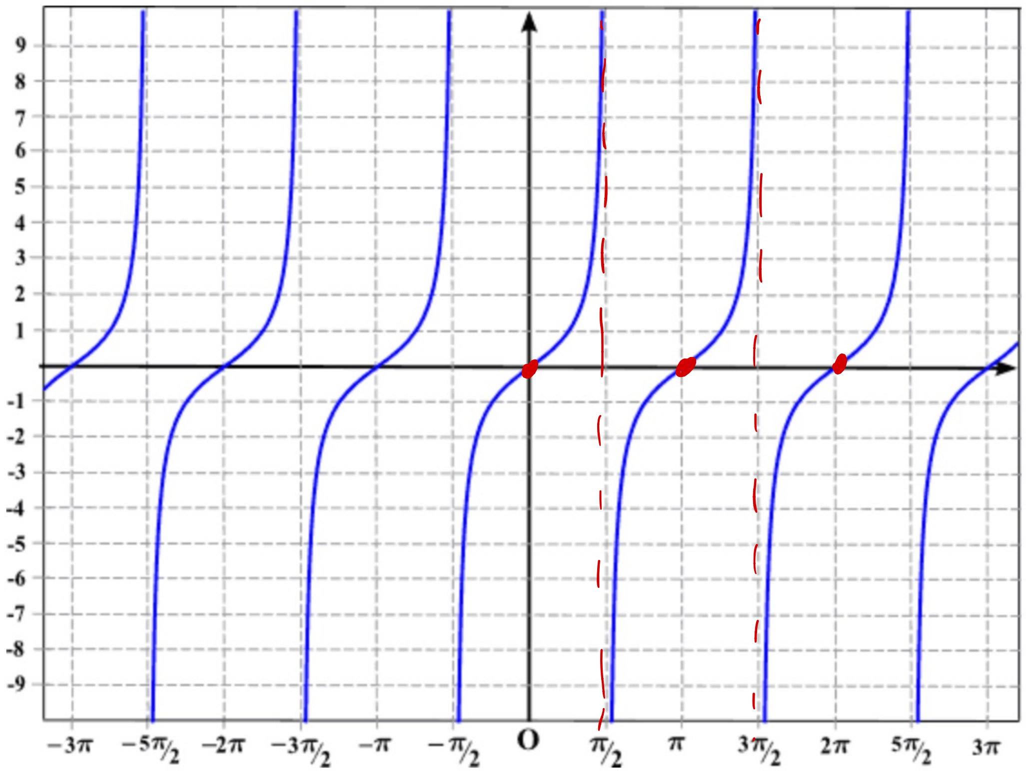


PERIODO 2π

$y = \cos(x)$

$\cos(x) : \mathbb{R} \rightarrow [-1, 1]$

$\cos(x) = 0 \quad \forall\, x = \frac{\pi}{2} + k\pi \,, \quad k \in \mathbb{Z}$



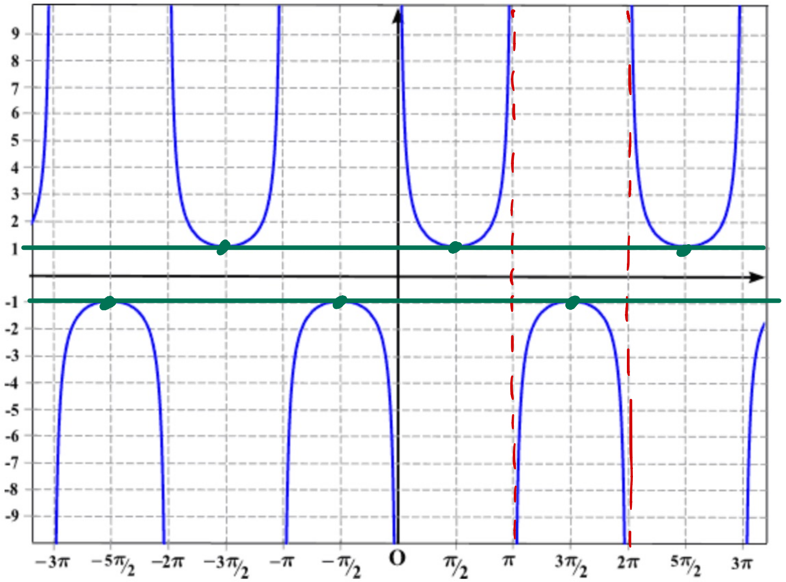
$$y = \tan(x) := \frac{\sin(x)}{\cos(x)}$$

$\tan(x) : \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi \right\} \rightarrow \mathbb{R}$

$x \neq \frac{\pi}{2} + k\pi \quad \left(\frac{\pi}{2} \,, \frac{3}{2}\pi \,, \frac{5}{2}\pi \,\ldots \right)$

$\tan(x) = 0 \quad \forall\, x = k\pi \quad k \in \mathbb{Z}$

GRAFICI FUNZIONI RECIPROCHE



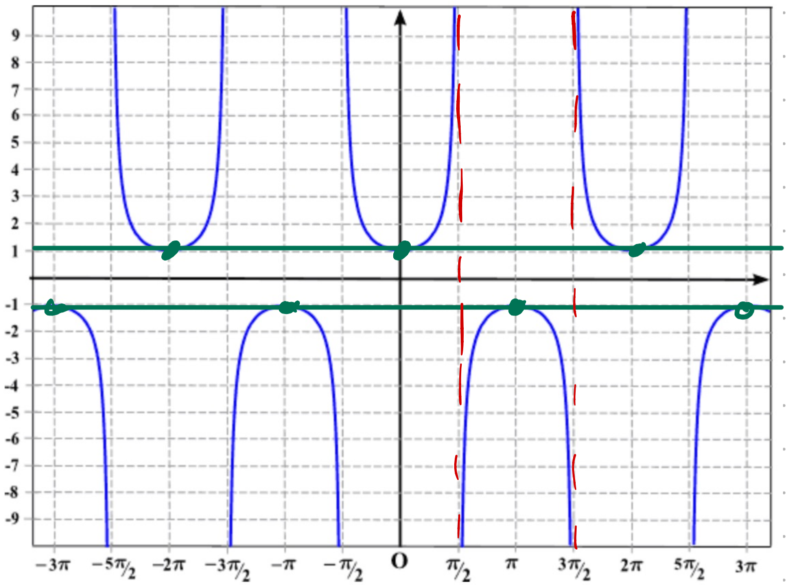
$$y = \operatorname{cosec}(x) := \frac{1}{\operatorname{sen}(x)}$$

$\operatorname{cosec}(x) : \mathbb{R} - \{k\pi\} \to \mathbb{R} - (-1, 1)$

$x \neq k\pi \quad k \in \mathbb{Z}$

$-1 \leq \operatorname{sen}(x) \leq 1 \Rightarrow \frac{1}{\operatorname{sen}(x)} \geq 1 \vee \frac{1}{\operatorname{sen}(x)} \leq -1$

$\operatorname{cosec}(x) = \pm 1 \quad \forall x = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$



$$y = \sec(x) := \frac{1}{\cos(x)}$$

$\sec(x) : \mathbb{R} - \{\frac{\pi}{2} + k\pi\} \to \mathbb{R} - (-1, 1)$

$x \neq \pi/2 + k\pi \quad k \in \mathbb{Z}$

$y \geq 1 \vee y \leq -1$

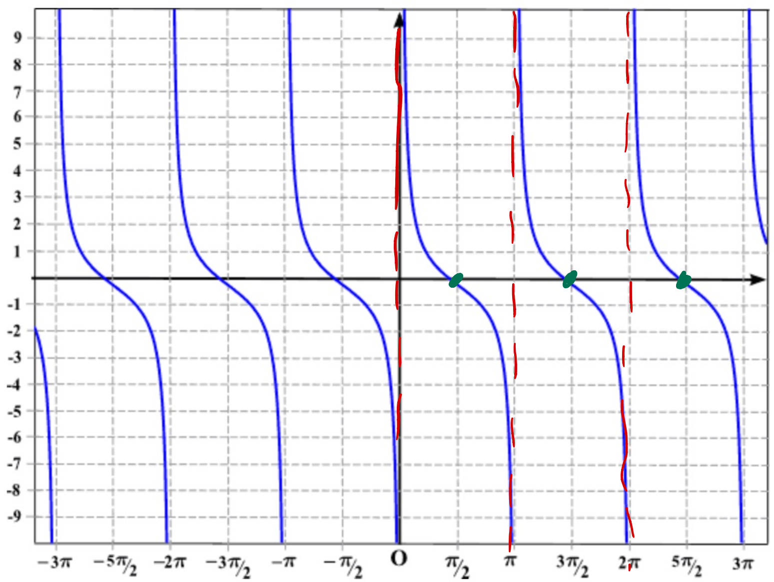
$\sec(x) = \pm 1$

$\hookrightarrow \forall x = k\pi \quad k \in \mathbb{Z}$

$$y = \cot(x) := \frac{1}{\tan(x)}$$

$\cot x : \mathbb{R} - \{k\pi\} \to \mathbb{R}$

$x \neq k\pi \quad k \in \mathbb{Z}$



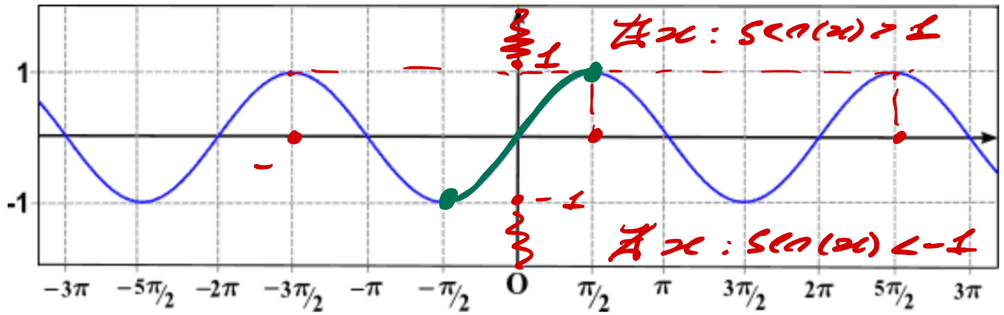
GRAFICI FUNZIONI INVERSE

F è invertibile $\iff$ biunivoca $\iff$ f è iniettiva e suriettiva

$f : D \subseteq \mathbb{R} \to \mathbb{R}$

$f : x \to y$ è iniettiva $\iff \forall x_1, x_2 \in D : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$f : x \to y$ è suriettiva $\iff \forall y \in \mathbb{R}, \exists x \in D : f(x) = y$

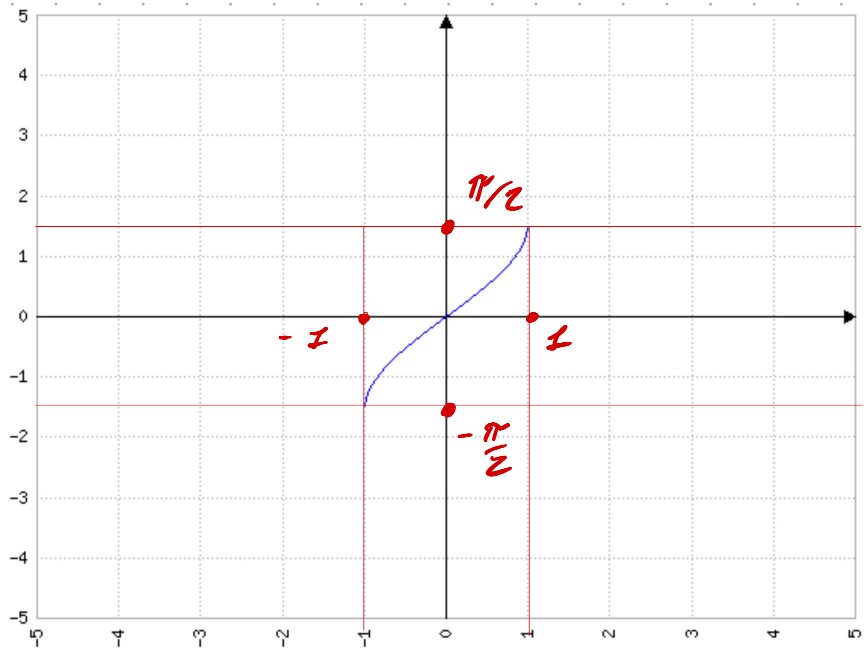


$y = sen(x)$ non è suriettiva $\Rightarrow$ restringiamo il codominio a $[-1, 1]$

se $y = 1$ $\exists \infty\, x \in \mathbb{R} : sen(x) = 1$

↳ non è iniettiva $\Rightarrow$ restringiamo a $[-\frac{\pi}{2}, \frac{\pi}{2})$

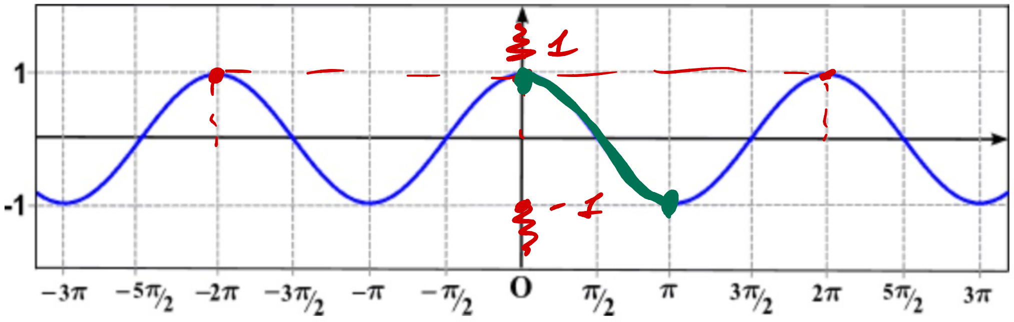
↳ prendiamo $sen(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$



$\Rightarrow$ $y = arcsin(x)$

$arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

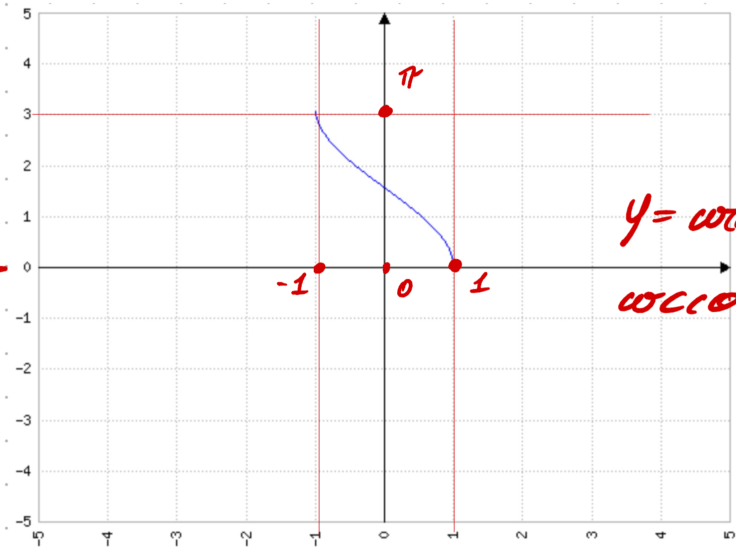
GRAFICI FUNZIONI INVERSE



↳ idem per il $\cos(x)$

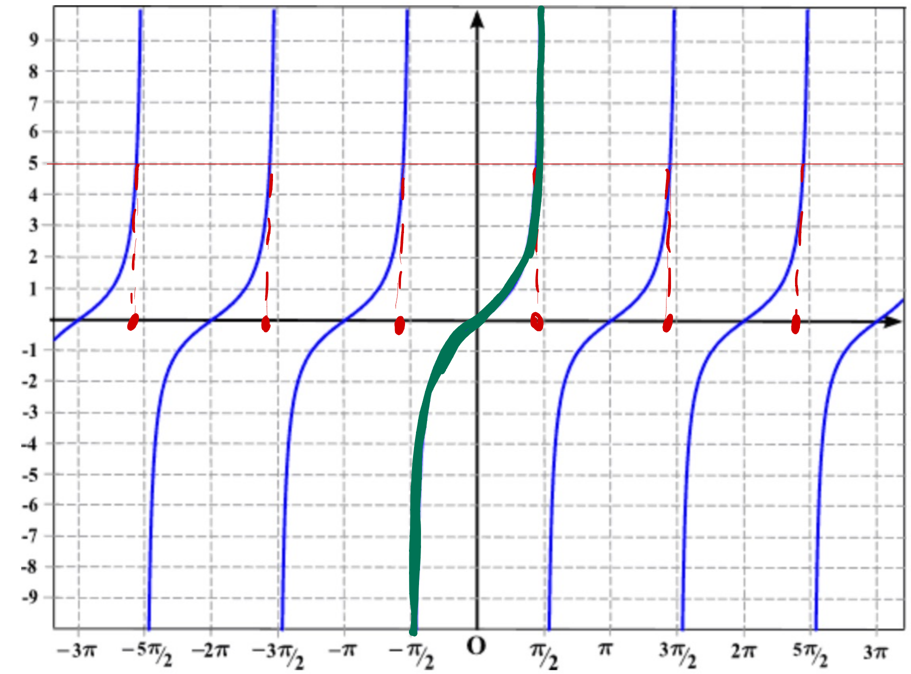
$\Rightarrow$ prendiamo $\cos(x) : [0, \pi]$

$\Rightarrow$

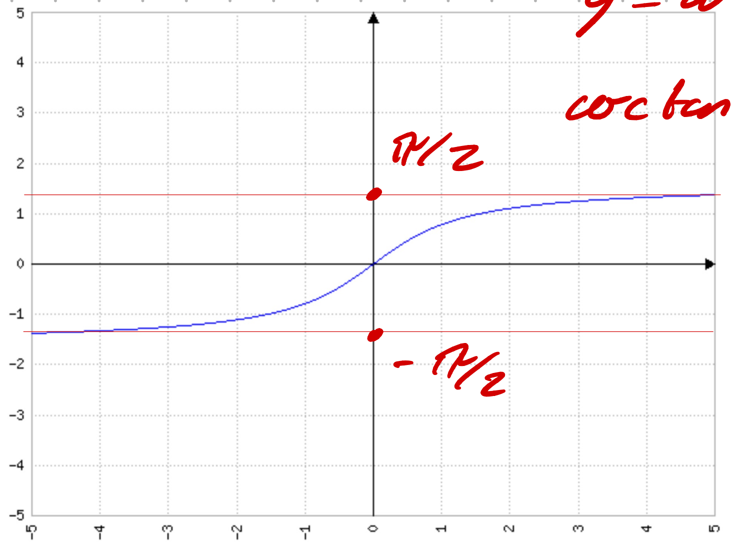


$y = \arccos(x)$

$\arccos : [-1, 1] \to [0, \pi]$



$\Rightarrow$



$y = \arctan(x)$

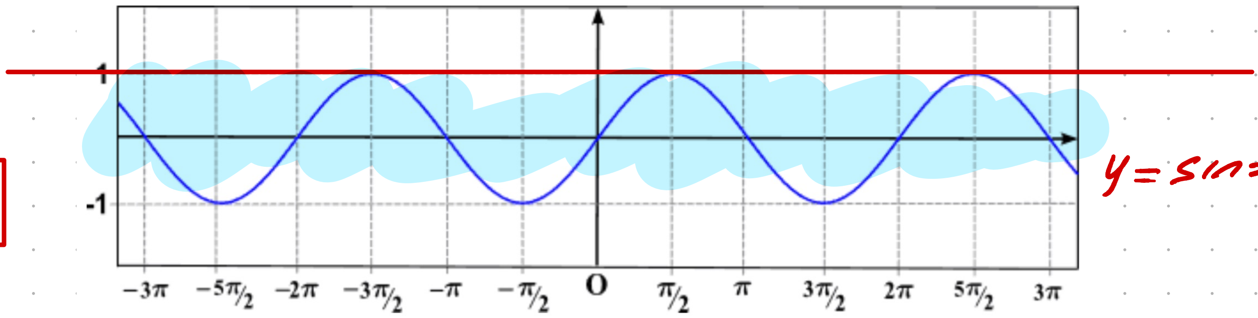
$\arctan : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\forall\, y \in \mathbb{R}\ \exists\, x \in D \subseteq \mathbb{R} : \tan(x) = y \Rightarrow$ è suriettiva $\to$ OK codominio $\mathbb{R}$

ma se $y = 1$ $\exists\, \infty\, x : \tan(x) = 1 \Rightarrow$ non è iniettiva $\to D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- $\sin x \le 1$

  ↳ $\boxed{\text{Infinite } x}$

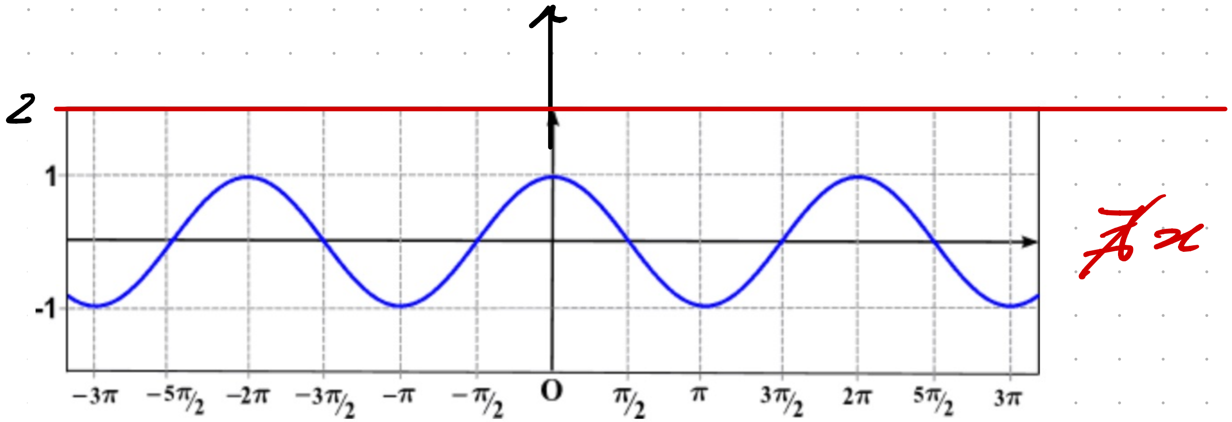


$y=1$

$y=\sin x \le 1 \ \forall x$

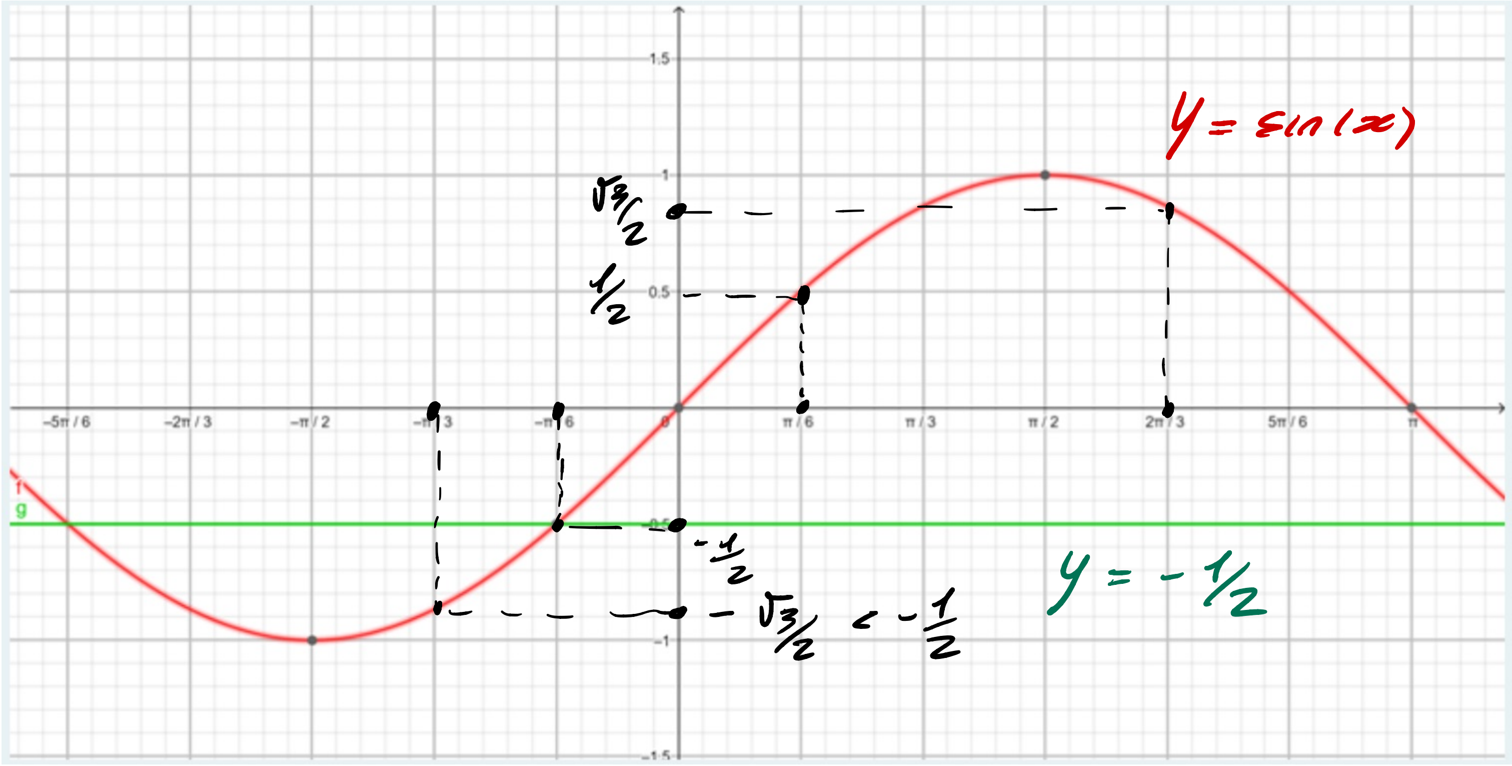
- $\cos x \ge 2$

  ↳ $\boxed{0}$

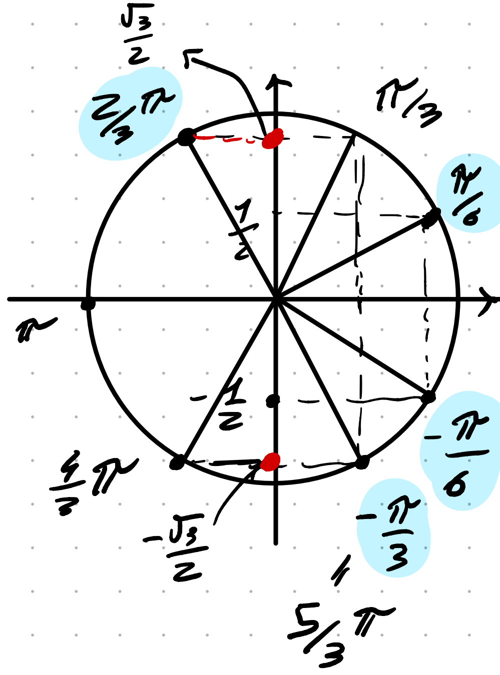


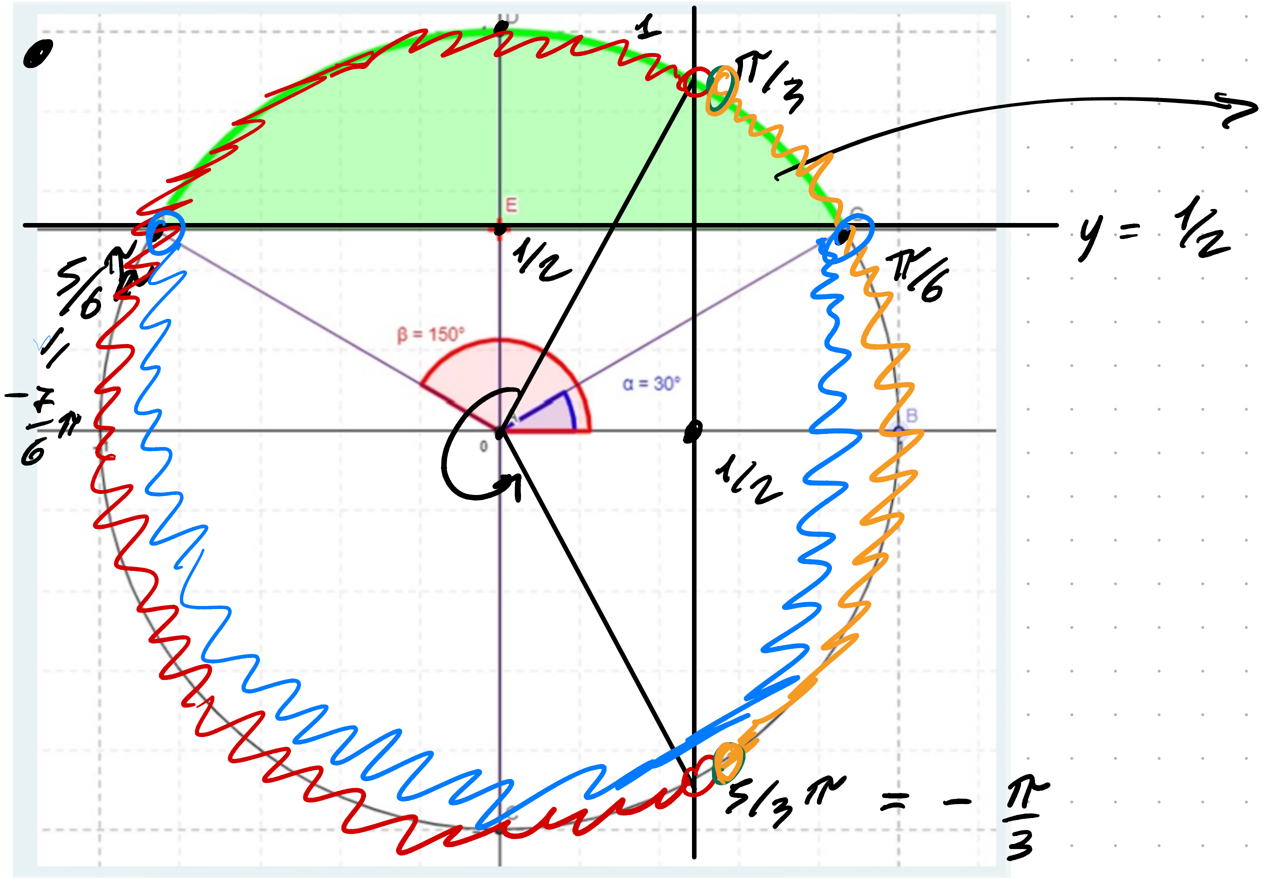
$y=2$

$\nexists x : y=\cos x \ge 2$



- ○ $\sin\left(-\frac{\pi}{6}\right) < -\frac{1}{2}$ $\rightarrow$ $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \Rightarrow$ NO!
- ○ $\sin\left(\frac{\pi}{6}\right) < -\frac{1}{2}$ $\rightarrow$ $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} > -\frac{1}{2} \Rightarrow$ NO!
- ~~○~~ $\sin\left(-\frac{\pi}{3}\right) < -\frac{1}{2}$ $\rightarrow$ $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} < -\frac{1}{2} \Rightarrow$ $\boxed{\text{SI!!!}}$
- ○ $\sin\left(\frac{2}{3}\pi\right) < \sin\left(\frac{5}{3}\pi\right)$ $\rightarrow$ $\sin\left(\frac{2}{3}\pi\right) > \sin\left(\frac{5}{3}\pi\right) \Rightarrow$ NO!





$$\frac{\pi}{6} + 2k\pi < x < \frac{5}{6}\pi + 2k\pi$$

- ○ $\cos x < \frac{1}{2}$ → $\pi/3 + 2k\pi < x < 5/3 + 2k\pi$ → NO!
- ○ $\cos x > \frac{1}{2}$ → $-\frac{\pi}{3} + 2k\pi < x < \frac{\pi}{3} + 2k\pi$ → NO!
- ○ $\sin x < \frac{1}{2}$ → $-\frac{7}{6}\pi + 2k\pi < x < \frac{\pi}{6} + 2k\pi$ → NO!
- ⊗ $\sin x > \frac{1}{2}$ → $\frac{\pi}{6} + 2k\pi < x < \frac{5}{6}\pi + 2k\pi$ → $\boxed{\text{SI!!}}$

- $\cos x < \frac{1}{2}$ $\quad x \in [0, 2\pi] \rightarrow \boxed{\frac{\pi}{3} < x < \frac{5}{3}\pi}$

$\hookrightarrow$ 1 GIRO

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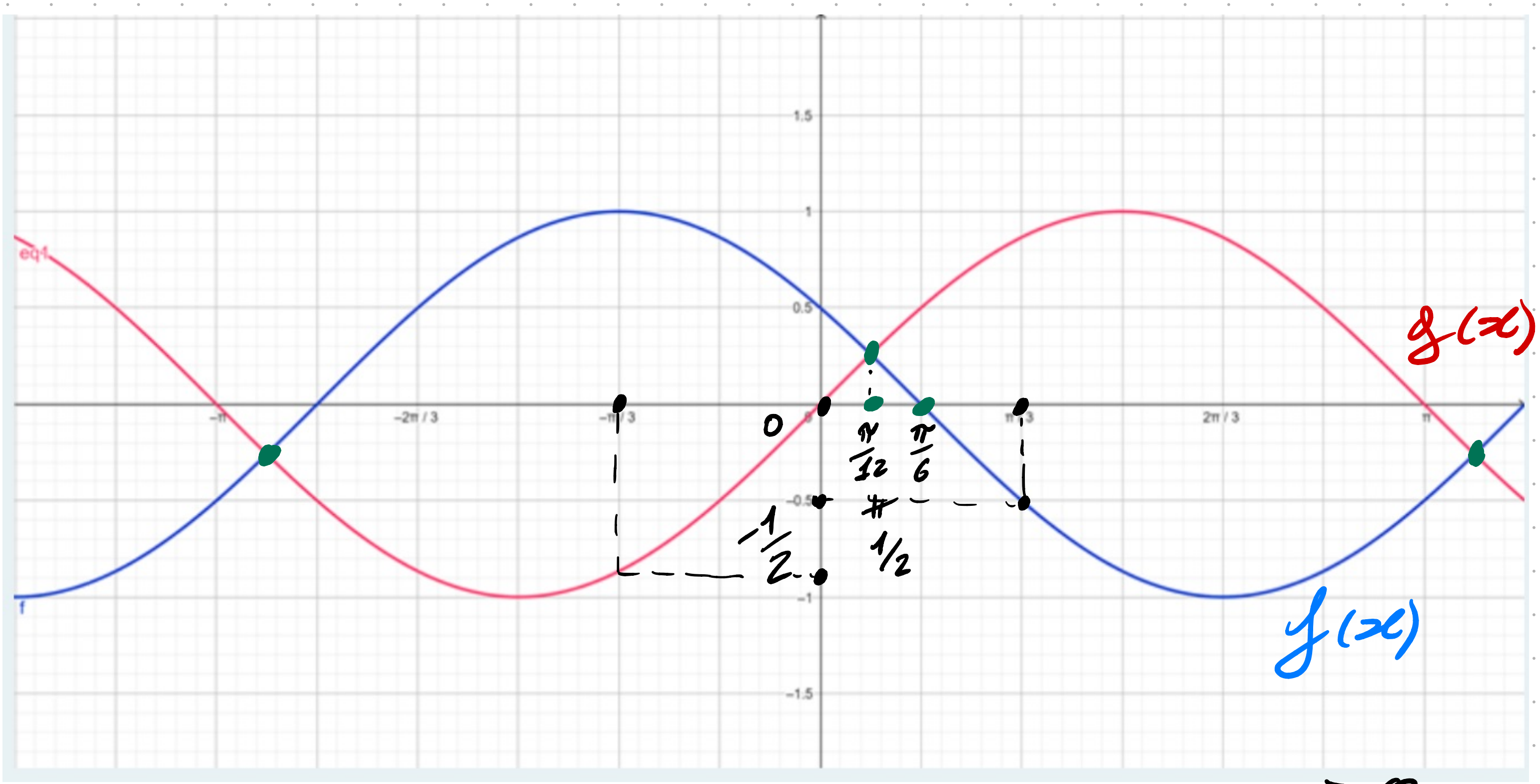


$g(x) = \sin(x)$

$f(x) = \sin(x \pm c)$

oppure

$f(x) = \cos(x \pm c)$

- $f(x) = g(x)$ se $x = \frac{1}{2}$ $\Rightarrow$ NO
- $f\left(\frac{\pi}{3}\right) > -g\left(-\frac{\pi}{3}\right)$ $\Rightarrow f\left(\frac{\pi}{3}\right) = \underbrace{-\frac{1}{2}}_{<0} > \overbrace{-\left(\underbrace{g\left(-\frac{\pi}{3}\right)}_{<0}\right)}^{>0} \Rightarrow f\left(\frac{\pi}{3}\right) < -\left(g\left(-\frac{\pi}{3}\right)\right) \Rightarrow$ NO
- $f(x) = \cos\left(x - \frac{\pi}{3}\right)$ $\Rightarrow x = -\frac{\pi}{3} \rightarrow \cos\left(-\frac{2}{3}\pi\right) \neq 1 \Rightarrow$ NO !!!
- $f\left(\frac{\pi}{6}\right) = g(0)$ $\Rightarrow f\left(\frac{\pi}{6}\right) = 0, \; g(0) = 0 \Rightarrow \boxed{\text{SI!!!}}$

$f(x)$? $\rightarrow$ $\cos\left(-\frac{\pi}{3} \pm c\right) = 1 \Rightarrow -\frac{\pi}{3} \pm c = 0 \Rightarrow c = \frac{\pi}{3} \Rightarrow f(x) = \cos\left(x + \frac{\pi}{3}\right)$

VERIFICA (1) $f\left(\frac{\pi}{6}\right) = 0$, (2) $\operatorname{sen} x = \cos\left(x + \frac{\pi}{3}\right)$ in $x = \frac{\pi}{12}$

1) $f\left(\frac{\pi}{6}\right) = 0 \rightarrow \cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi + 2\pi}{6}\right) = \cos\left(\frac{3}{6}\pi\right) = \cos\frac{\pi}{2} = 0$

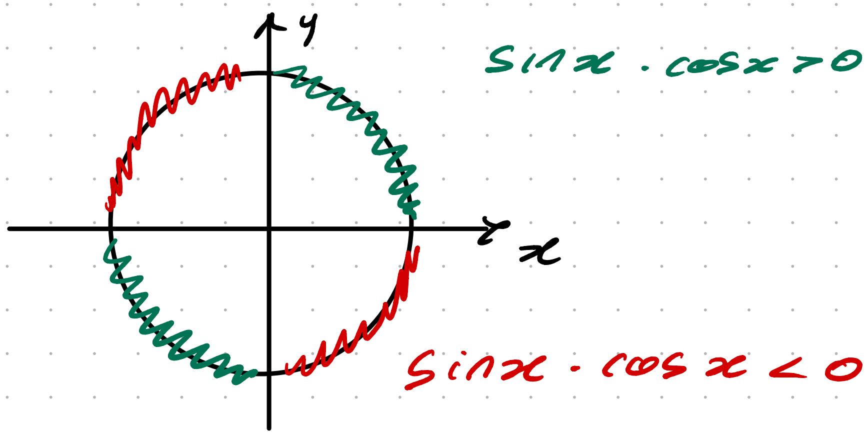
2) $\operatorname{sen} x = \cos\left(x + \frac{\pi}{3}\right) \Rightarrow \cos\left(\frac{\pi}{2} - x\right) = \cos\left(x + \frac{\pi}{3}\right)$

$\uparrow \cos\left(\pi/2 - \theta\right) = \sin\theta$

↳ $\frac{\pi}{2} - x = x + \frac{\pi}{3} \rightarrow 2x = \frac{\pi}{2} - \frac{\pi}{3} \rightarrow 2x = \frac{3\pi - 2\pi}{6} = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{12}$

- $\sin(x) \cos(x) > 0 \quad \forall x \in \mathbb{R}$

↳ $\boxed{\text{FALSO}}$

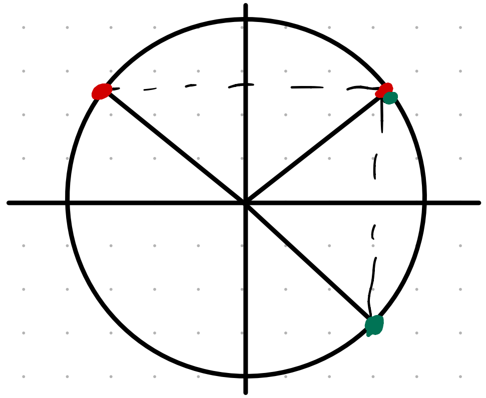


- $\sin 2x = \sin\left(\frac{\pi}{3} - x\right)$

$$2x = \frac{\pi}{3} - x + 2k\pi \rightarrow 3x = \frac{\pi}{3} + 2k\pi$$

$$\rightarrow \boxed{x = \frac{\pi}{9} + \frac{2}{3}k\pi}$$

$$2x = \pi - \frac{\pi}{3} + x + 2k\pi \rightarrow \boxed{x = \pi - \frac{\pi}{3} + 2k\pi}$$



- $\cos 3x = \cos 2x$

$$3x = 2x + 2k\pi \rightarrow x = 2k\pi$$

$$3x = -2x + 2k\pi \rightarrow x = \frac{2}{5}k\pi$$

- $\cos x + 4 - \dfrac{3}{\cos x + 2} = 0$

$(\cos x + 2)\cos x + 4(\cos x + 2) - 3 = 0$

$\hookrightarrow \cos^2 x + 2\cos x + 4\cos x + 8 - 3 = 0$

$\hookrightarrow \cos^2 x + 6\cos x + 5 = 0$ $\qquad t = \cos x$

$\hookrightarrow t^2 + 6t + 5 = 0$

$(t+5)(t+1) = 0 \longrightarrow t_1 = -5, \quad t_2 = -1$

$\Rightarrow \cos x = -5 \rightarrow$ IMPOSSIBILE

$\cos x = -1 \rightarrow x = \pi + 2k\pi$

• $\sin x + \cos x - \sin x \cdot \cos x = 1$

$$\hookrightarrow \begin{cases} \sin x + \cos x - \sin x \cdot \cos x = 1 \\ \sin^2 x + \cos^2 x = 1 \end{cases} \qquad \sin x = p \,,\ \cos x = q$$

$$\begin{cases} p + q - p \cdot q = 1 \\ p^2 + q^2 = 1 \end{cases} \rightarrow \begin{cases} p + q - p \cdot q = 1 \\ (p+q)^2 - 2p \cdot q = 1 \end{cases} \rightarrow \begin{cases} p + q = 1 + p \cdot q \\ (p+q)^2 - 2p \cdot q = 1 \end{cases}$$

$$\begin{cases} p + q = 1 + p \cdot q \\ (1 + p \cdot q)^2 - 2pq = 1 \end{cases} \rightarrow \begin{cases} p + q = 1 + p \cdot q \\ \cancel{1} + p^2 q^2 + \cancel{2pq} - \cancel{2pq} = \cancel{1} \end{cases}$$

$$\begin{cases} p + q = 1 + p \cdot q \\ p^2 \cdot q^2 = 0 \end{cases} \Rightarrow \begin{cases} p = 0 \rightarrow q = 1 \\ q = 0 \rightarrow p = 1 \end{cases}$$

$$\Rightarrow (\sin x = 0 \ \wedge \ \cos x = 1) \vee (\sin x = 1 \ \wedge \ \cos x = 0)$$

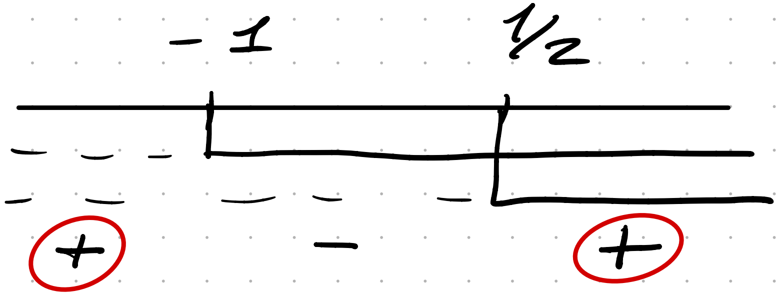
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$(x = 2k\pi) \quad \vee \quad \left(x = \frac{\pi}{2} + 2k\pi\right)$$

- $2\cos^2 x + \cos x - 1 > 0$ $\qquad t = \cos x$

$$2t^2 + t - 1 > 0$$

$$2t^2 + t - 1 = 0 \iff t_{1/2} = \frac{-1 \pm \sqrt{1+8}}{4} = \begin{cases} \dfrac{-1-3}{4} = -1 \\[2mm] \dfrac{-1+3}{4} = \dfrac{1}{2} \end{cases}$$



$$\Rightarrow (t < -1) \vee \left(t > \frac{1}{2}\right)$$

$\cos x < -1 \rightarrow$ IMPOSSIBILE

$\cos x > 1/2$

$$\left(0 < x < \frac{\pi}{3}\right) \vee \left(\frac{5}{3}\pi < x < 2\pi\right)$$

oppure

$$\left(-\frac{\pi}{3} < x < \frac{\pi}{3}\right)$$