LA FANTASTICA STORIA DEL PRODE ERONE NEL MAGICO MONDO DEI POLIGONI CIRCOLARI

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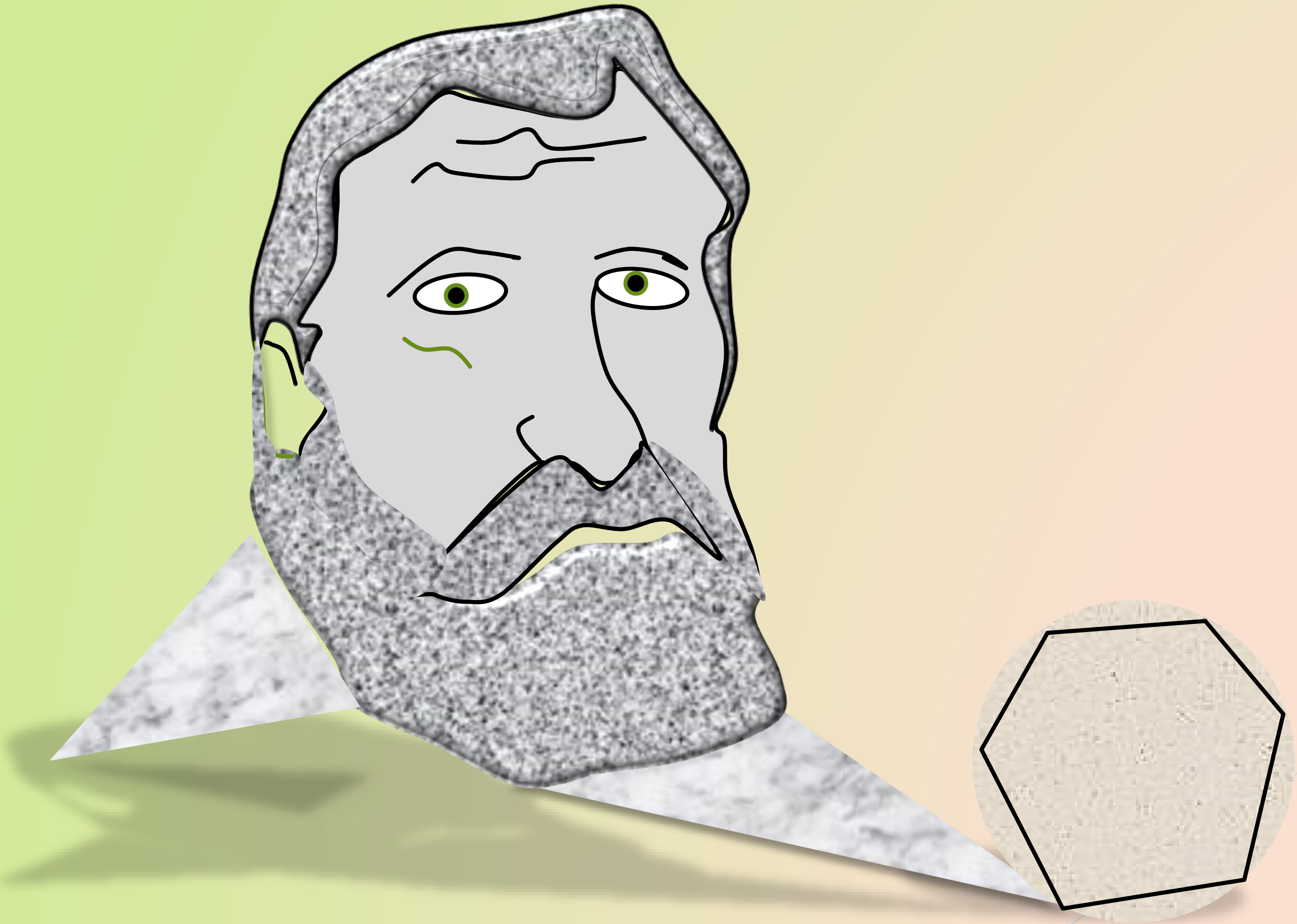
Politecnico di Milano



Seminari FDS
02/12/2020

POLITECNICO
MILANO 1863

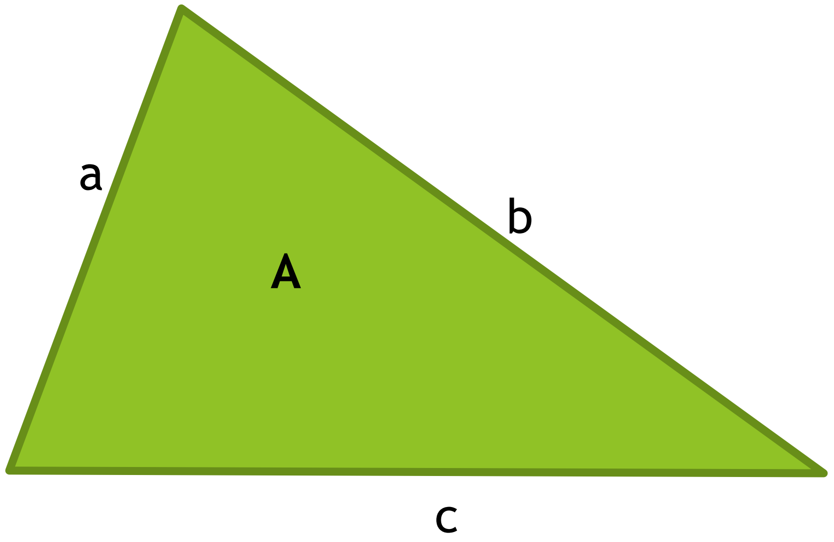
CI FACCIAMO DOMANDE

- *Cos'è la Formula di Erone per il triangolo?*
- *Ma Pitagora c'entra?*
- *E' possibile estendere la formula di Erone a quadrilateri?*
- *E a poligoni con più di 4 lati?*
- *Ma sono proprio i lati i veri protagonisti?*

PER CERCARE RISPOSTE

- Precisare
- Sfaccettare
- Generalizzare
- Correggere
- Esplorare

LA FORMULA DI ERONE

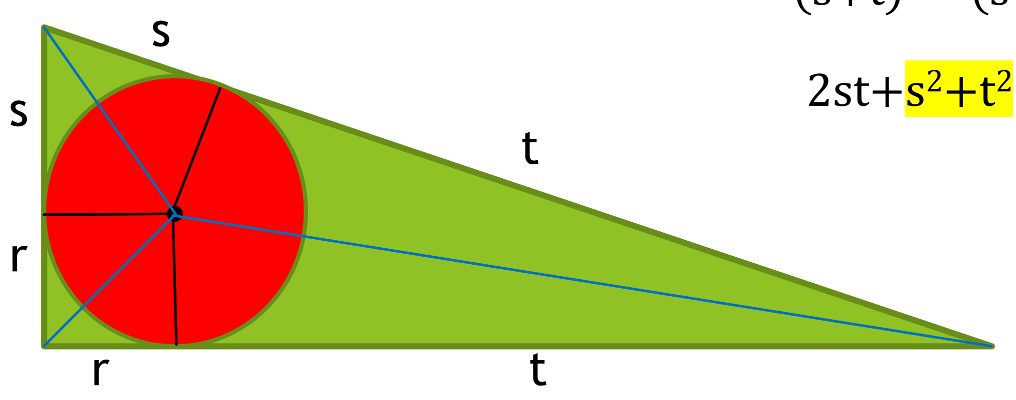


2p=a+b+c

$$A^2 = p(p-a)(p-b)(p-c)$$

IL LEMMA DI LAENG

- Partiamo da un triangolo rettangolo



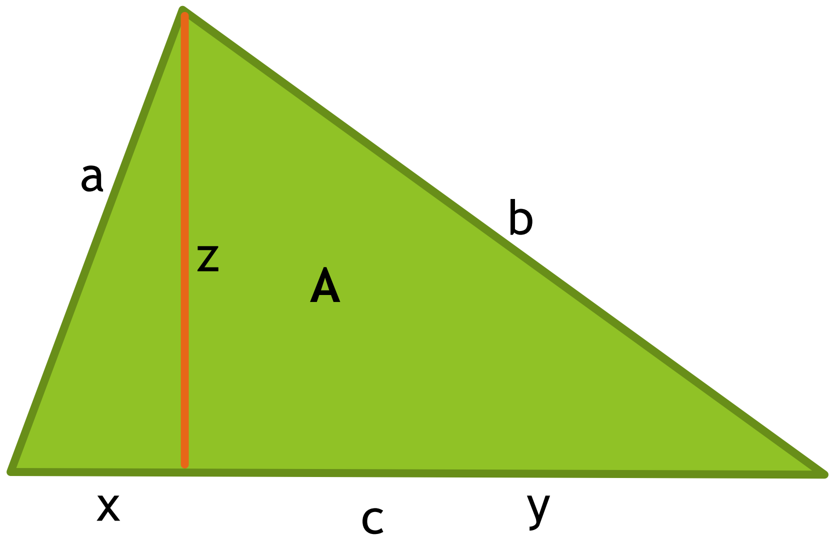
$(s+t)^2 = (s+r)^2 + (r+t)^2$ **Teorema di Pitagora**

$2st + s^2 + t^2 = 2r(r+s+t) + s^2 + t^2$

- $A = \frac{1}{2}[(r+s)r + (s+t)r + (r+t)r] = r(r+s+t)$
- $A = \frac{1}{2}[(r+s)(r+t)] \Rightarrow (r+s)(r+t) = 2r(r+s+t) \quad \Rightarrow st = r(r+s+t) = A$
- $A^2 = rst(r+s+t)$ **Formula di Erone per i triangoli rettangoli**

IL DOPPIO PITAGORA

- Grazie al Teorema di Pitagora abbiamo



$$c^2 = (x+y)^2 = x^2 + y^2 + 2xy$$

$$(c^2 - x^2 - y^2)^2 = 4x^2y^2$$

$$x^2 = a^2 - z^2$$ **Pitagora 1**

$$y^2 = b^2 - z^2$$ **Pitagora 2**

$$(c^2 - a^2 - b^2 + 2z^2)^2 = 4(a^2 - z^2)(b^2 - z^2)$$

$$4z^2c^2 = 4a^2b^2 - (c^2 - a^2 - b^2)^2$$

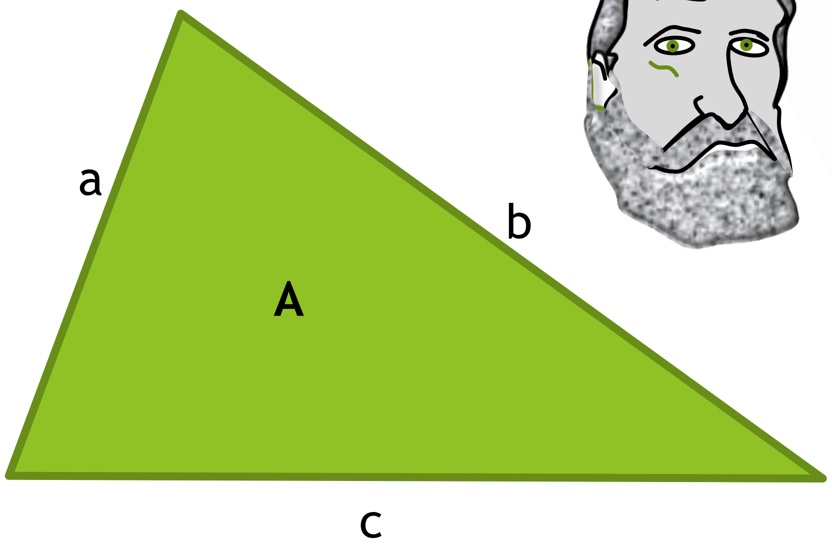
$$16A^2 = \left(2ab - (c^2 - a^2 - b^2)\right)\left(2ab + (c^2 - a^2 - b^2)\right)$$

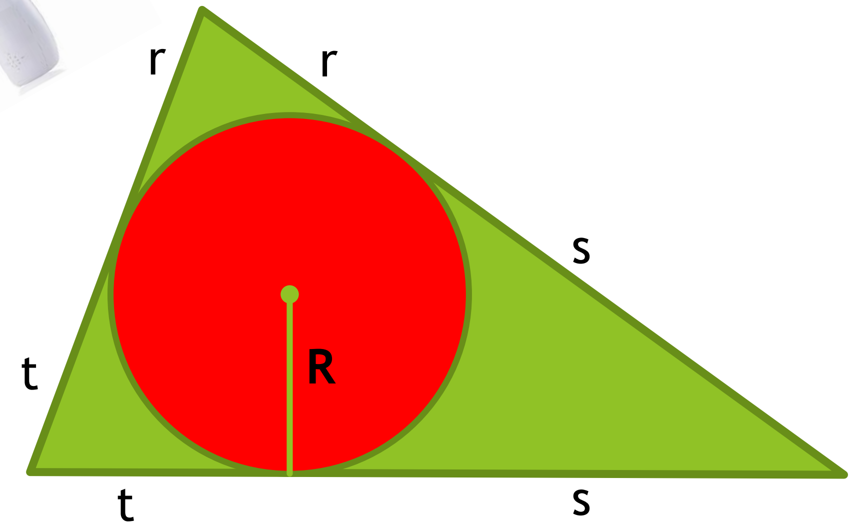
$$16A^2 = \left((a+b)^2 - c^2\right)\left(c^2 - (a-b)^2\right)$$

$$16A^2 = (a+b+c)(a+b-c)(a+c-b)(c+b-a)$$

$$16\mathrm{A}^2 = 16\mathrm{p}(\mathrm{p}-\mathrm{c})(\mathrm{p}-\mathrm{b})(\mathrm{p}-\mathrm{a})$$ **Formula di Erone**

ERONE RIVISITATO





$$A^2 = \mathrm{p}(\mathrm{p}-\mathrm{c})(\mathrm{p}-\mathrm{b})(\mathrm{p}-\mathrm{a})$$

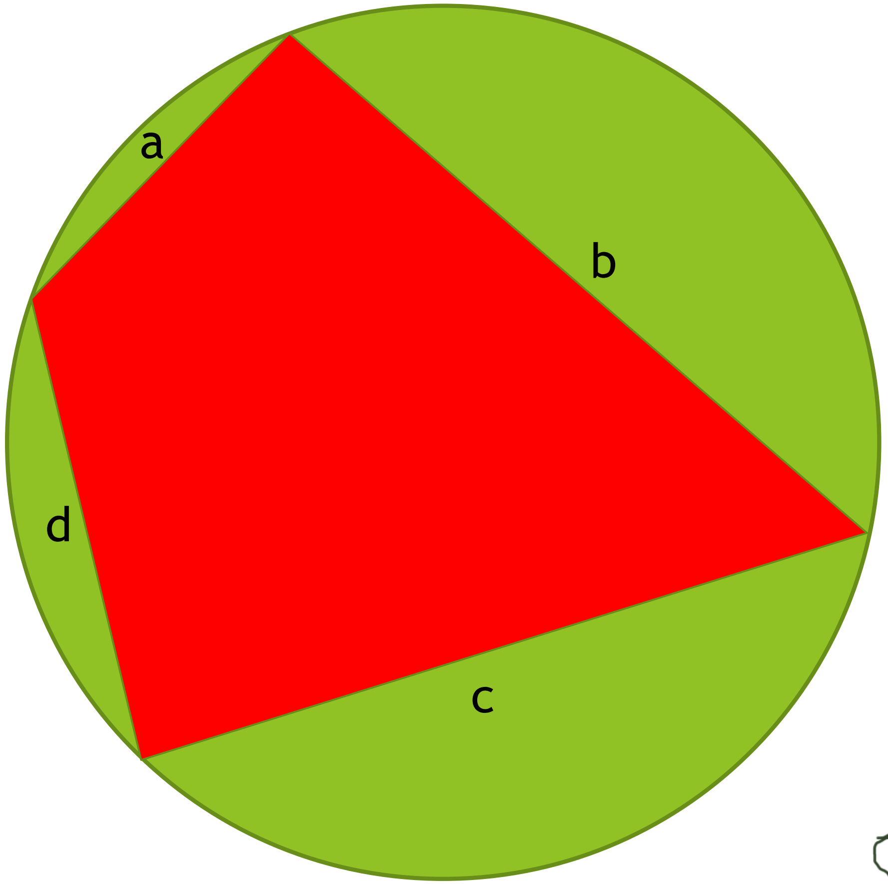
$$A^2 = (r+s+t)rst = prst$$

OSSERVAZIONE

$$A = pR \Longrightarrow R = \frac{A}{p} = \sqrt{\frac{rst}{r+s+t}}$$

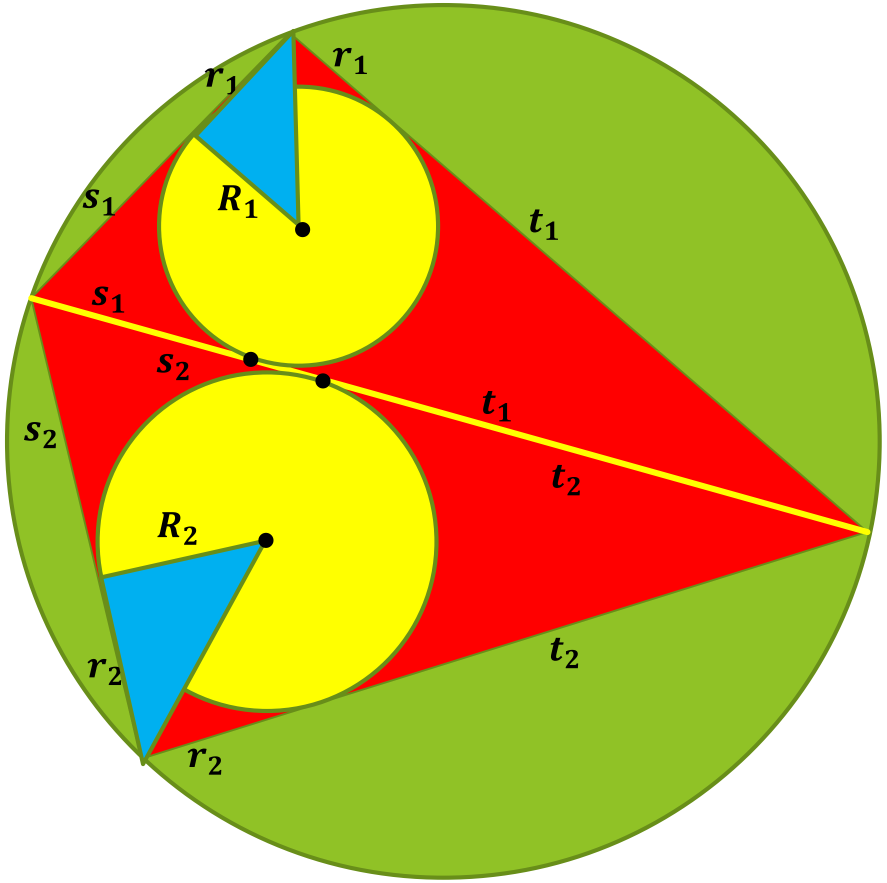
BRAHMAGUPTA

$$2p = a + b + c + d$$



$$A^2 = (p-a)(p-b)(p-c)(p-d)$$

COME SI DIMOSTRA?



$2p = a + b + c + d$
$= 2\mathrm{r}_1 + 2\mathrm{r}_2 + \mathrm{s}_1 + \mathrm{t}_1 + \mathrm{s}_2 + \mathrm{t}_2$

$p = \mathrm{r}_1 + \mathrm{r}_2 + \mathrm{s}_1 + \mathrm{t}_1 = p_1 + r_2$

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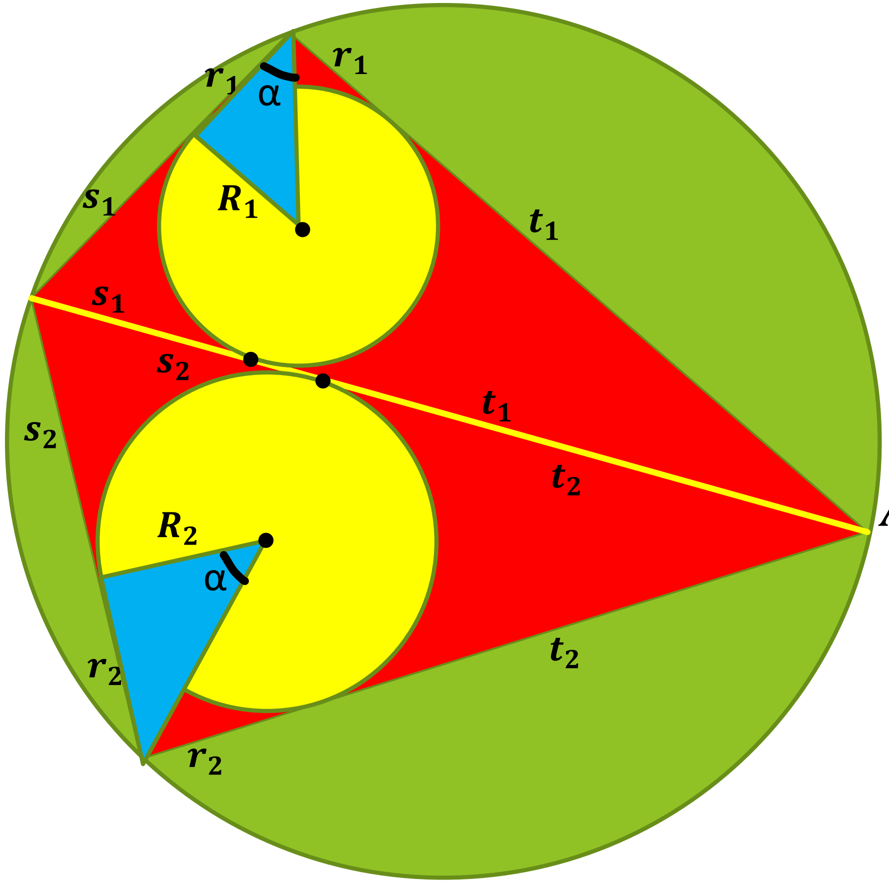

$$\frac{R_1}{r_1} = \frac{r_2}{R_2} \implies R_1 R_2 = r_1 r_2$$

$$p_1 R_1^2 p_2 R_2^2 = r_1 s_1 t_1 r_2 s_2 t_2$$

$$A_1 A_2 = p_1 R_1 p_2 R_2 = s_1 t_1 s_2 t_2$$

$$R = \sqrt{\frac{rst}{r+s+t}}$$

$$A^2 = (A_1 + A_2)^2 = A_1^2 + A_2^2 + 2A_1A_2 =$$

$$= r_1 s_1 t_1 (r_1 + s_1 + t_1) + r_2 s_2 t_2 (r_2 + s_2 + t_2) +$$
$$+ \mathrm{p_1 p_2 R_1 R_2} + \mathrm{s_1 t_1 s_2 t_2} =$$

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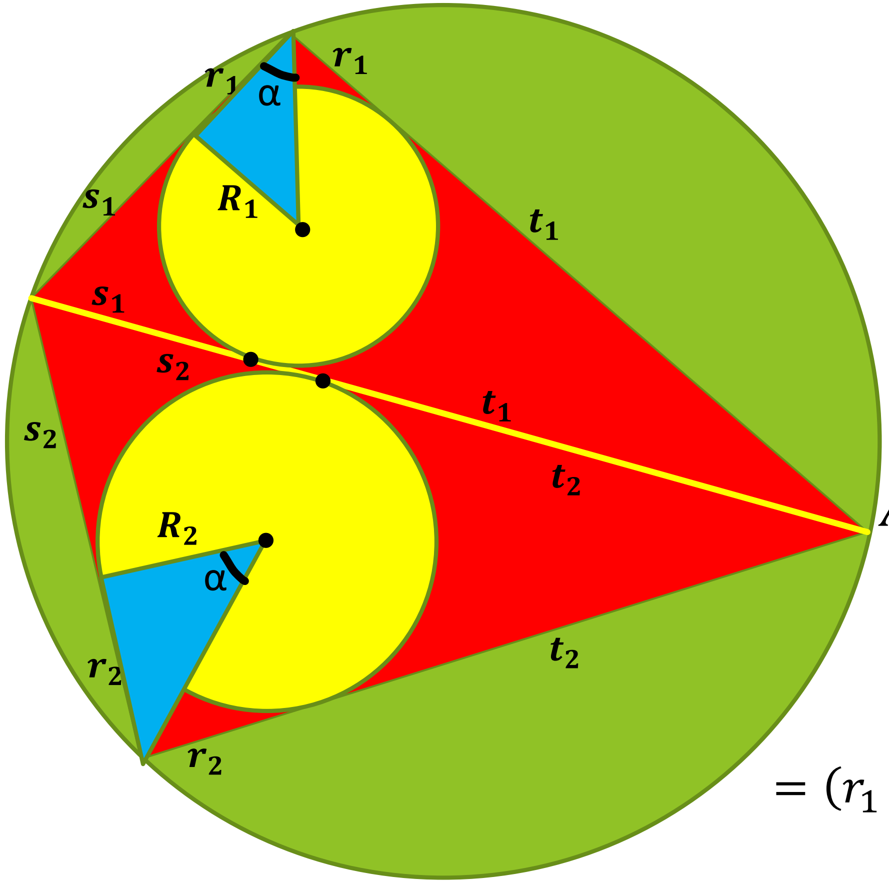

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$$= (r_1 + s_1 + t_1) r_1 s_1 t_1 + (r_2 + s_2 + t_2) r_2 s_2 t_2 + +\boldsymbol{r_1 r_2 (r_1 + s_1 + t_1)(r_2 + s_2 + t_2)} + s_1 t_1 s_2 t_2 =$$

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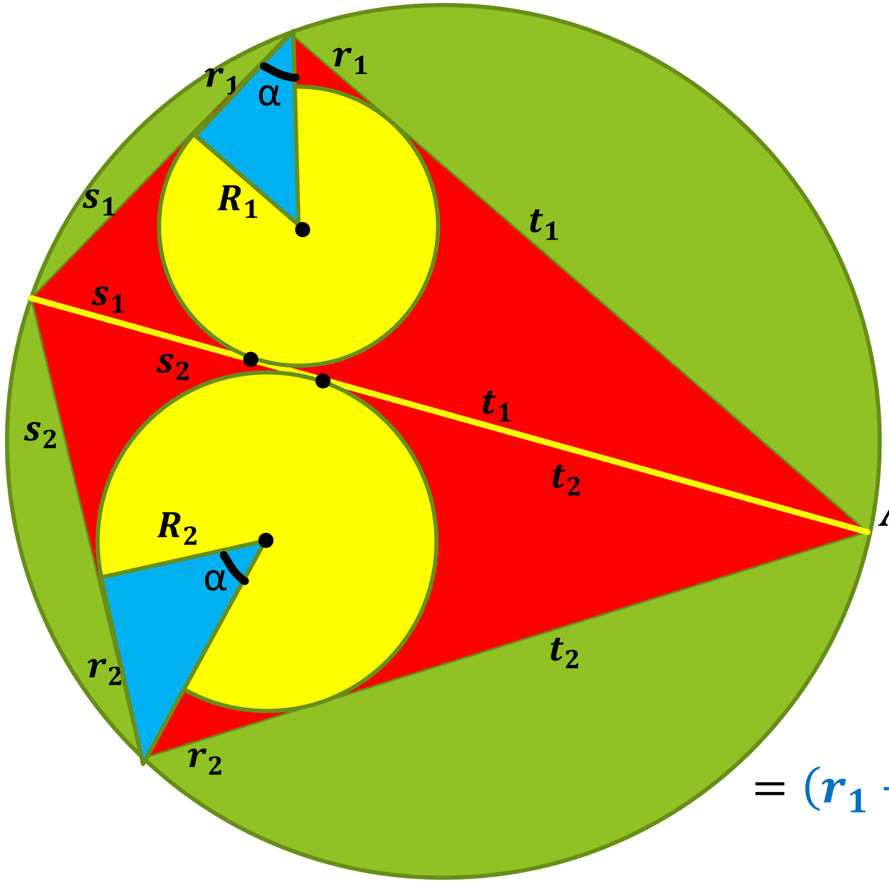

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$$= (r_1 + s_1 + t_1) r_1 s_1 t_1 + (r_2 + s_2 + t_2) r_2 s_2 t_2 +$$
$$+ r_1 r_2 (r_1 + s_1 + t_1)(r_2 + s_2 + t_2) + s_1 t_1 s_2 t_2 =$$

$$= r_1 \, (r_1 + s_1 + t_1)(r_2(r_2 + s_2 + t_2) + s_1 t_1) + \; s_2 t_2 (\mathrm{r_2}(\mathrm{r_2} + \mathrm{s_2} + \mathrm{t_2}) + \mathrm{s_1 t_1})$$

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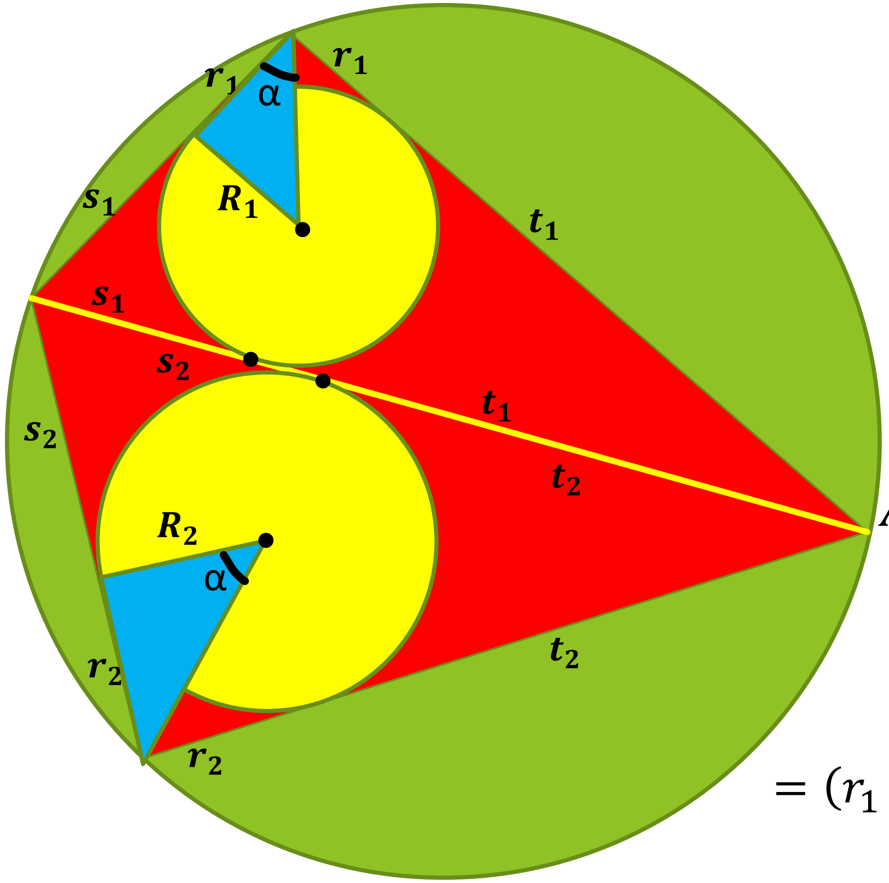

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$$= \boldsymbol{(r_2(r_2 + s_2 + t_2) + s_1 t_1)} (r_1 (r_1 + s_1 + t_1) + s_2 t_2) =$$

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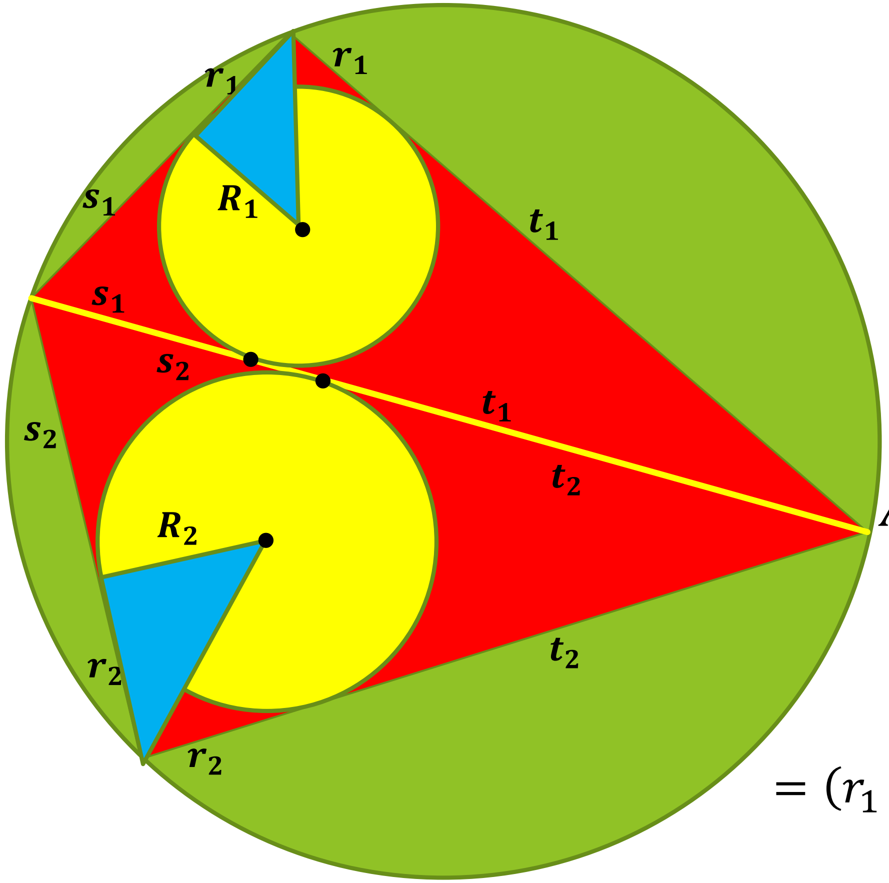

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$$= \mathrm{r_1}\ (\mathrm{r_1} + \mathrm{s_1} + \mathrm{t_1})(\mathrm{r_2}(\mathrm{r_2} + \mathrm{s_2} + \mathrm{t_2}) + \mathrm{s_1 t_1}) + \ \mathrm{s_2 t_2}(\mathrm{r_2}(\mathrm{r_2} + \mathrm{s_2} + \mathrm{t_2}) + \mathrm{s_1 t_1})$$

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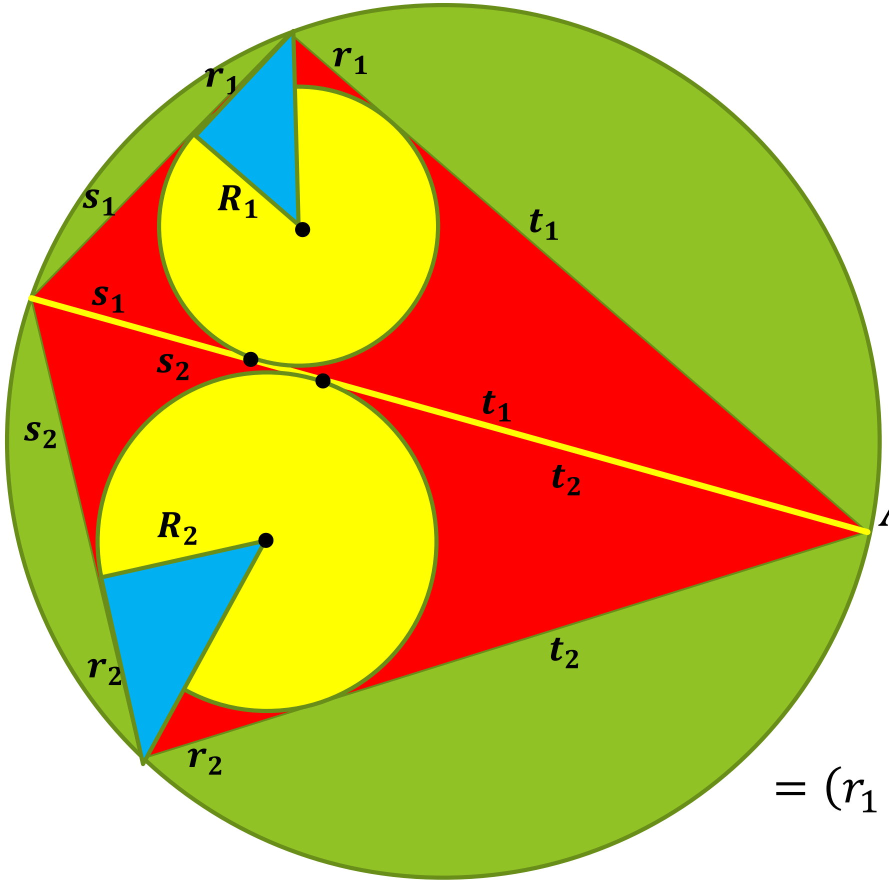

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$$= (r_2(r_2 + \boldsymbol{s_1} + \boldsymbol{t_1}) + s_1 t_1)(r_1(r_1 + \boldsymbol{s_2} + \boldsymbol{t_2}) + s_2 t_2) =$$

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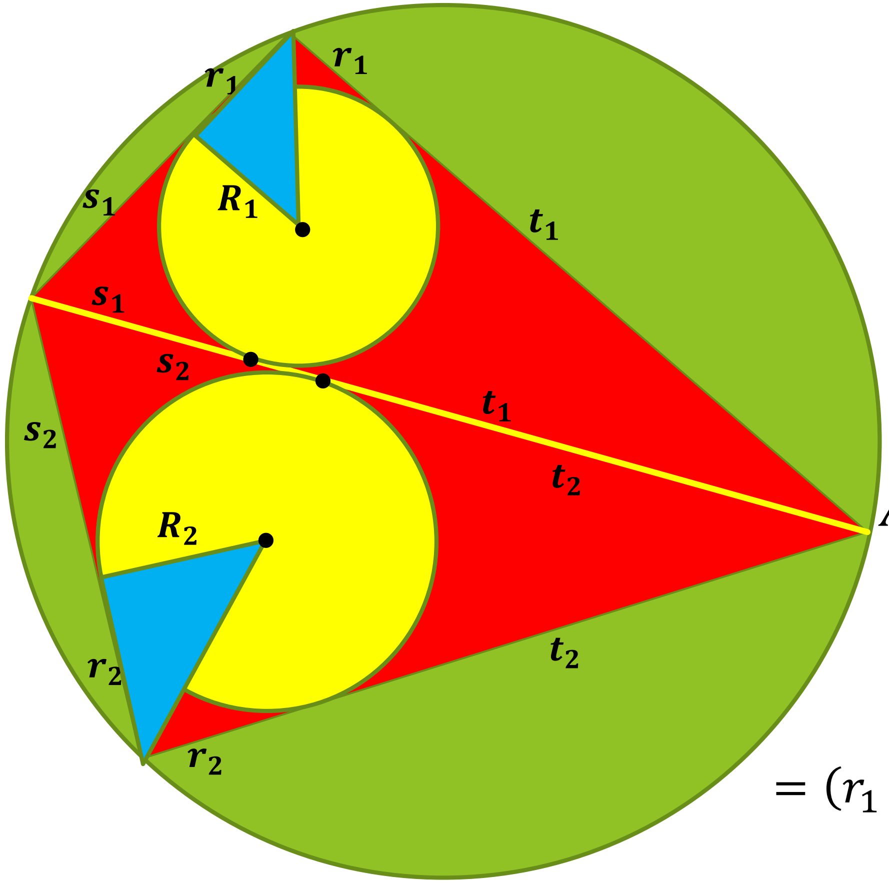

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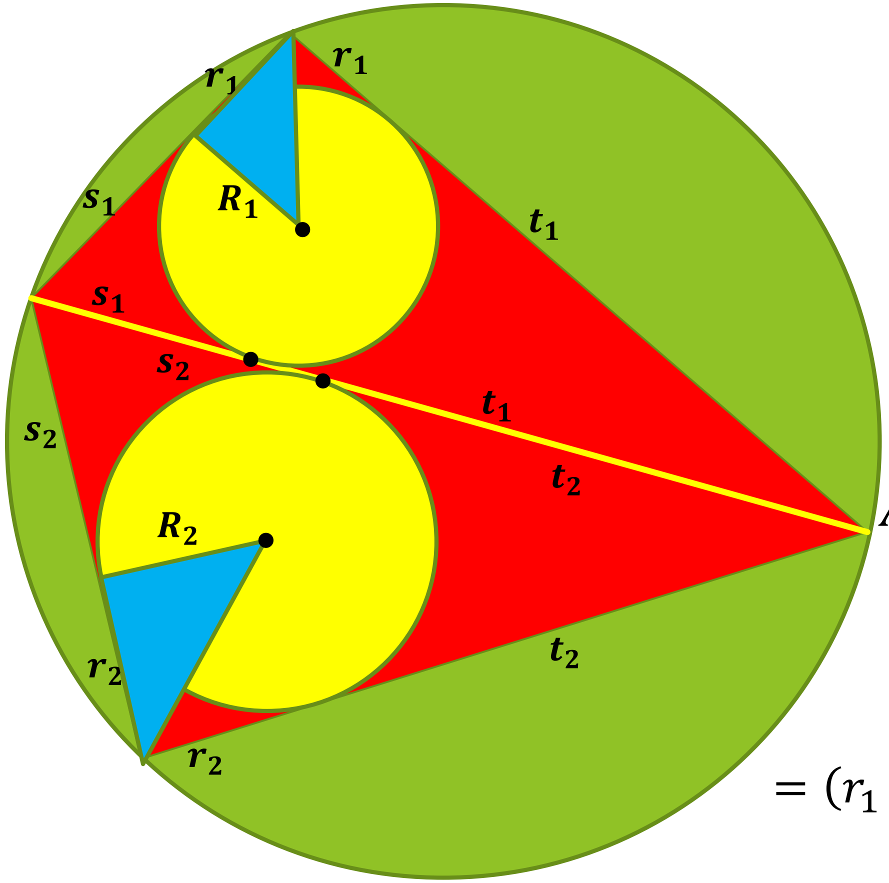
$$+ r_1 r_2 (r_1 + s_1 + t_1)(r_2 + s_2 + t_2) + s_1 t_1 s_2 t_2 =$$

$$= \mathrm{r_1\ (r_1 + s_1 + t_1)(r_2(r_2 + s_2 + t_2) + s_1 t_1) +\ s_2 t_2 (r_2 (r_2 + s_2 + t_2) + s_1 t_1)}$$

$$= (r_2(r_2 + s_1 + t_1) + s_1 t_1)(r_1(r_1 + s_2 + t_2) + s_2 t_2) =$$

$$= (r_2 + t_1)(r_2 + s_1)(r_1 + s_2)(r_1 + t_2)$$

COME SI DIMOSTRA?



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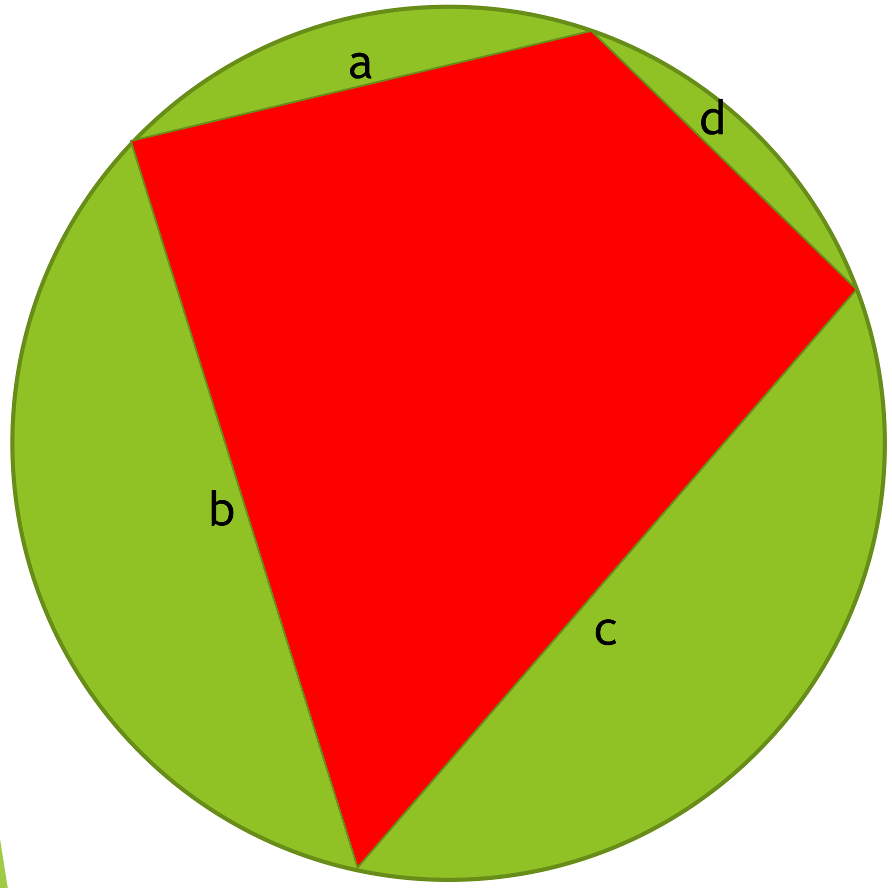
$$= (r_1 + s_1 + t_1) r_1 s_1 t_1 + (r_2 + s_2 + t_2) r_2 s_2 t_2 + \\ + r_1 r_2 (r_1 + s_1 + t_1)(r_2 + s_2 + t_2) + s_1 t_1 s_2 t_2 =$$

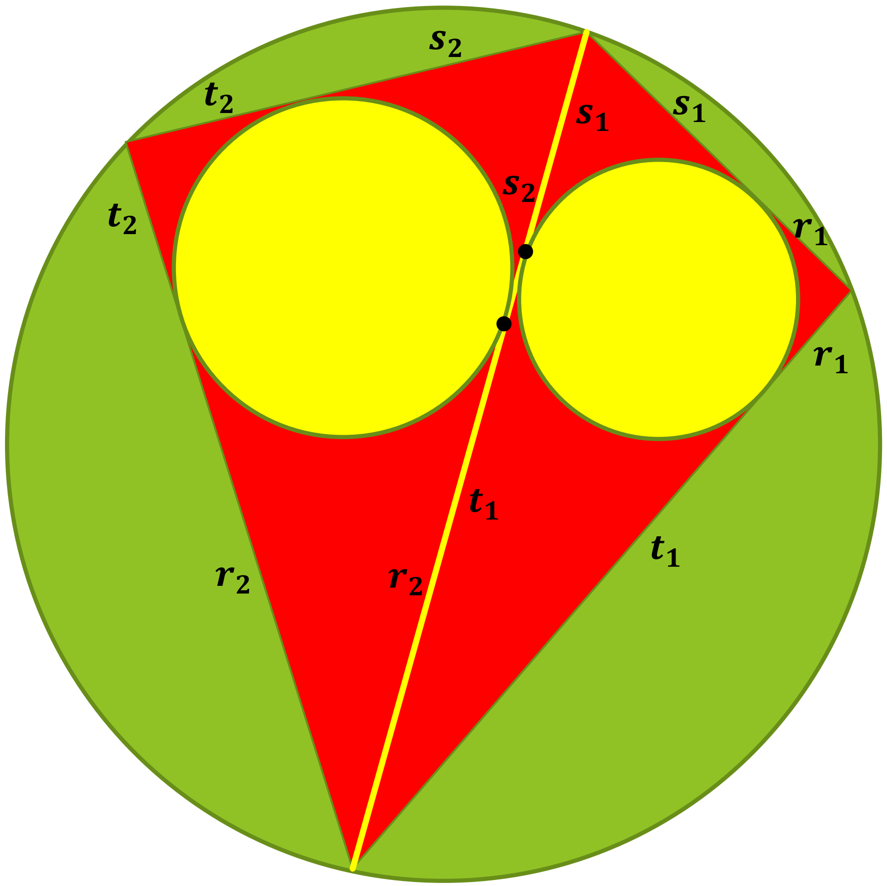
$$= \mathrm{r_1\ (r_1 + s_1 + t_1)(r_2(r_2 + s_2 + t_2) + s_1 t_1) +\ s_2 t_2 (r_2(r_2 + s_2 + t_2) + s_1 t_1)}$$

$$= (r_2(r_2 + s_1 + t_1) + s_1 t_1)(r_1(r_1 + s_2 + t_2) + s_2 t_2) =$$

$$= (r_2 + t_1)(r_2 + s_1)(r_1 + s_2)(r_1 + t_2) = (p-a)(p-b)(p-c)(p-d)$$

BRAHMAGUPTA RIVISITATO

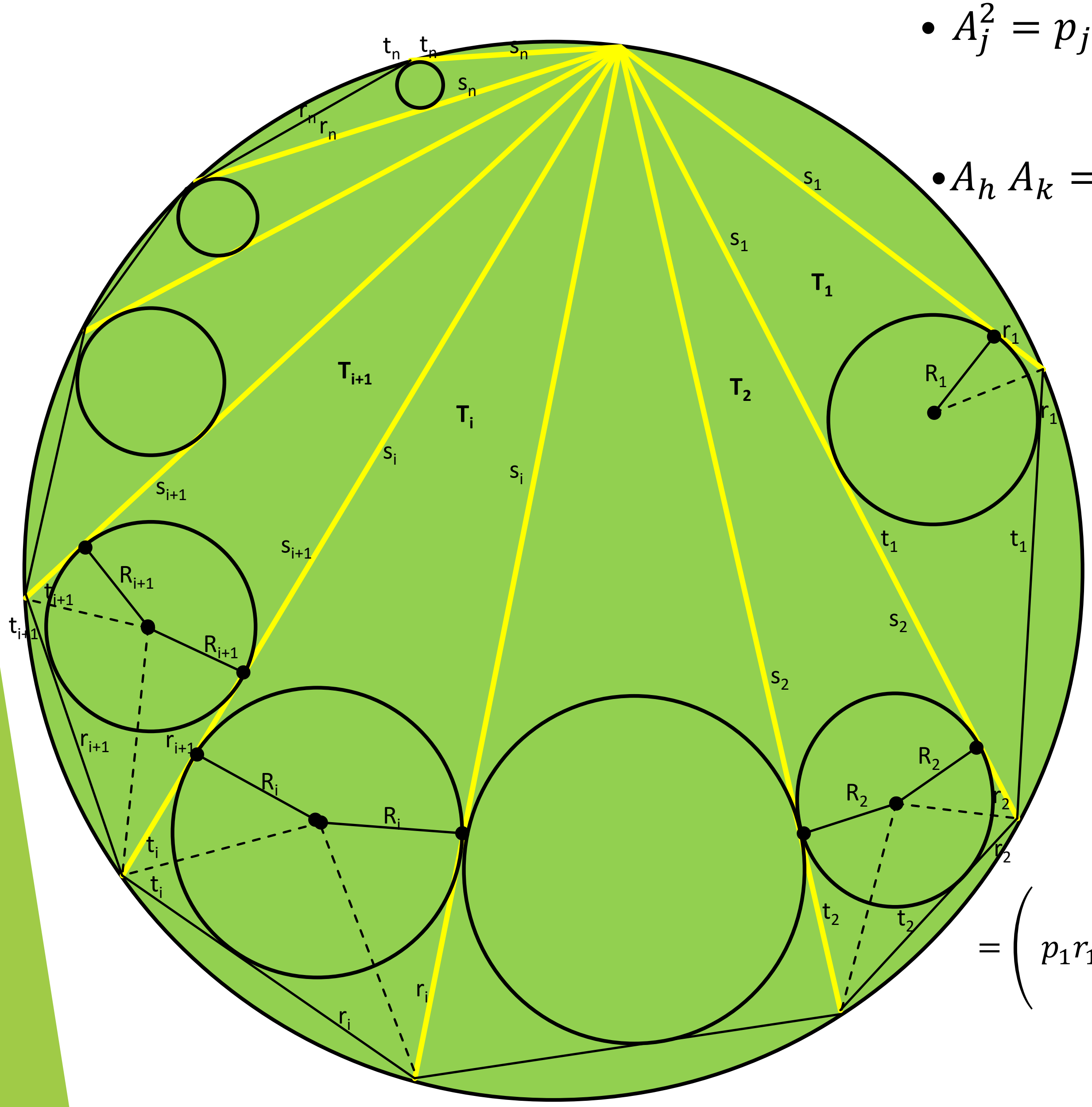




$$A^2 = (p-a)(p-b)(p-c)(p-d)$$

$$A^2 = (r_2+t_1)(r_2+s_1)(r_1+s_2)(r_1+t_2)$$
$$= (p_1 r_1 + s_2 t_2)(p_2 r_2 + s_1 t_1)$$

POLIGONI CICLICI



- $A_j^2 = p_j r_j s_j t_j$ per ogni $j \in \{1,2,\dots,n\}$
- $A_h\, A_k = s_h t_h s_k r_k \displaystyle\prod_{i=h+1}^{k-1} \frac{s_i}{p_i} = p_h r_h p_k t_k \prod_{i=h+1}^{k-1} \frac{p_i}{s_i}$

$$A(n)^2 = (A_1 + A_2 + \cdots + A_n)^2 =$$

$$= \quad \dots \quad =$$

$$= \left(p_1 r_1 + \sum_{q=2}^{n} r_q s_q \prod_{m=2}^{q-1} \frac{s_m}{p_m} \right)\left(s_1 t_1 + \sum_{q=2}^{n} p_q t_q \prod_{m=2}^{q-1} \frac{p_m}{s_m} \right)$$

OSSERVAZIONI E COMMENTI

$$A(n)^2 = \left(p_1 r_1 + \sum_{q=2}^{n} r_q s_q \prod_{m=2}^{q-1} \frac{s_m}{p_m} \right)\left(s_1 t_1 + \sum_{q=2}^{n} p_q t_q \prod_{m=2}^{q-1} \frac{p_m}{s_m} \right)$$

La formula fornisce il quadrato dell'area di un qualsiasi poligono ciclico come funzione simmetrica rispetto agli scambi $r_j \leftrightarrow t_j$ ed $s_j \leftrightarrow p_j$ per ogni $j \in \{1, \dots, n\}$.

Tutte le variabili coinvolte si ricavano dalle sole partizioni dei lati del poligono

$$\boldsymbol{A(1)^2 = (p_1 r_1 + 0)(s_1 t_1 + 0) = p_1 r_1 s_1 t_1}$$

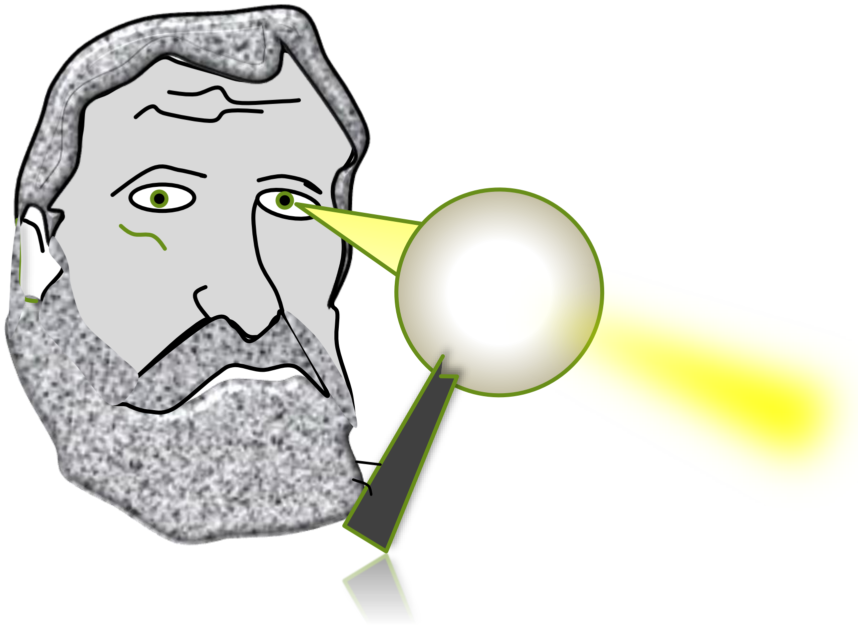
Erone

$$\boldsymbol{A(2)^2 = (p_1 r_1 + r_2 s_2)(s_1 t_1 + p_2 t_2) = (r_1 + r_2)(r_1 + s_2)(s_1 + t_2)(t_1 + t_2)}$$

Brahmagupta

$$\boldsymbol{A(3)^2 = \left(p_1 r_1 + r_2 s_2 + \frac{r_3 s_3 s_2}{p_2} \right)\left(s_1 t_1 + p_2 t_2 + \frac{p_3 t_3 p_2}{s_2} \right)}$$

$$\boldsymbol{A(4)^2 = \left(p_1 r_1 + r_2 s_2 + \frac{r_3 s_3 s_2}{p_2} + \frac{r_4 s_4 s_2 s_3}{p_2 p_3} \right)\left(s_1 t_1 + p_2 t_2 + \frac{p_3 t_3 p_2}{s_2} + \frac{p_4 t_4 P_3 p_3}{s_2 s_3} \right)}$$



GRAZIE PER LA VOSTRA ATTENZIONE