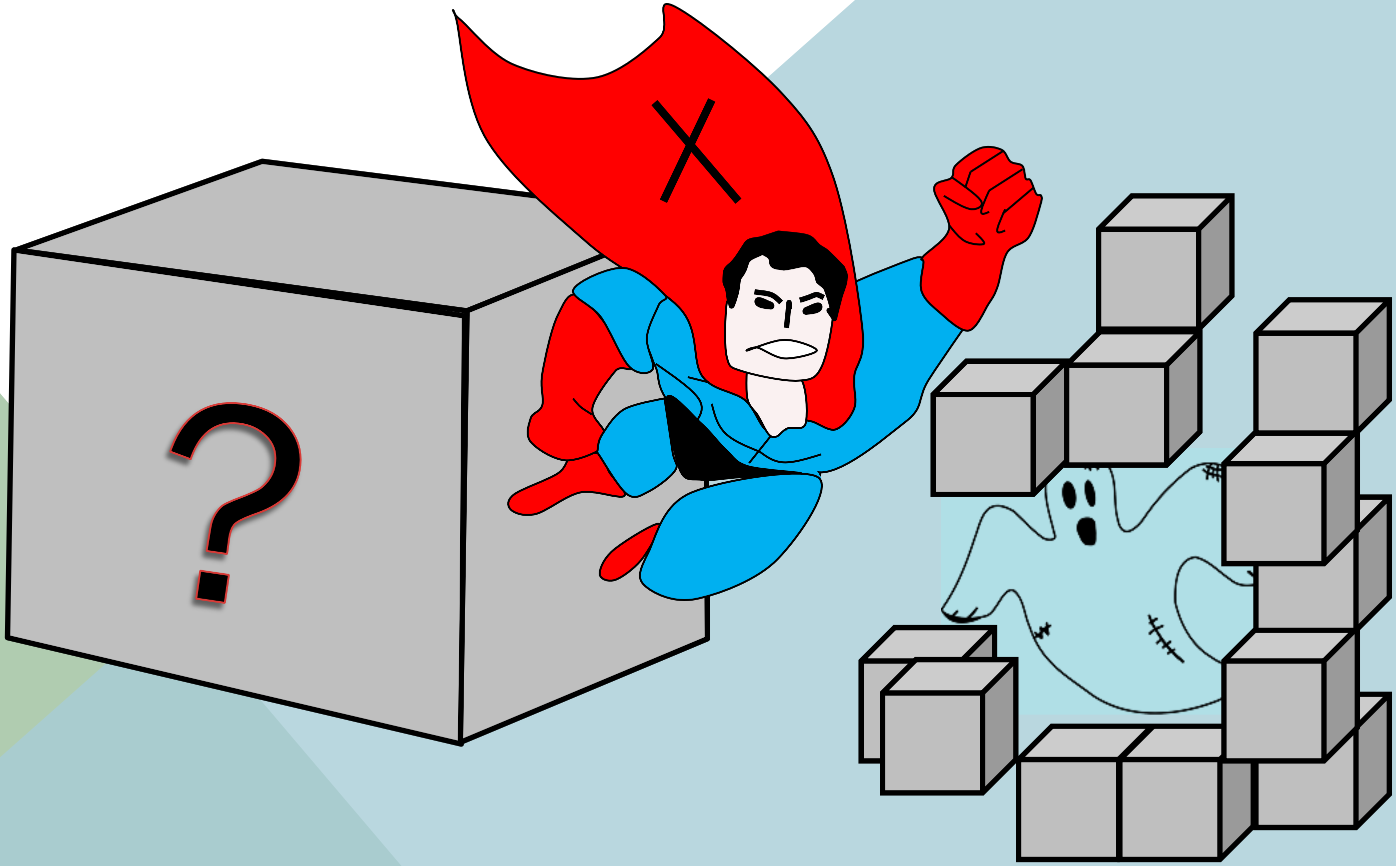
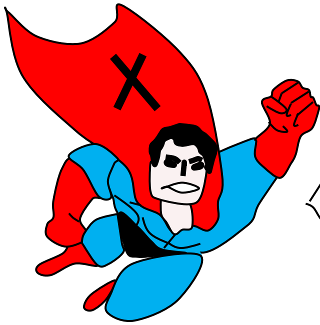
RICOSTRUIRE L'INVISIBILE...
FANTASMI PERMETTENDO

PAOLO DULIO

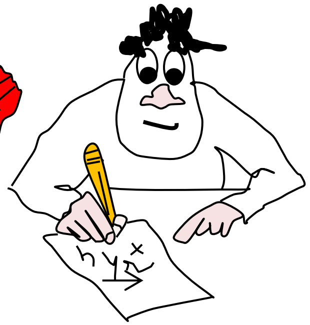
Dipartimento di Matematica Politecnico di Milano



CHARACTERS



MISTER X



Dr. Detector

Eric

Ghost

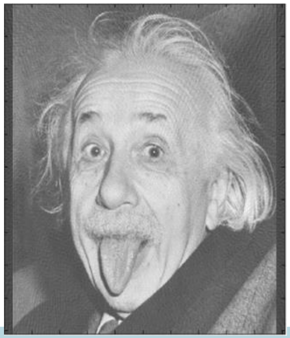
Dr. Raggio di Luna

Mr. Body

J. Radon

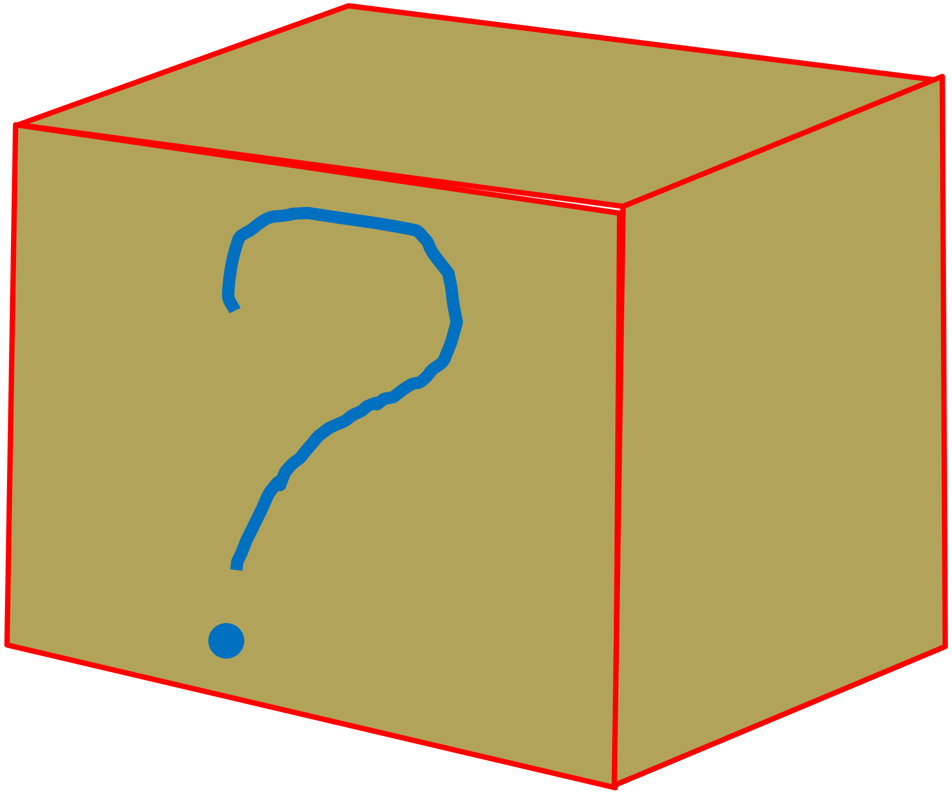
A. MacLeod Cormack

G. Newbold Hounsfield

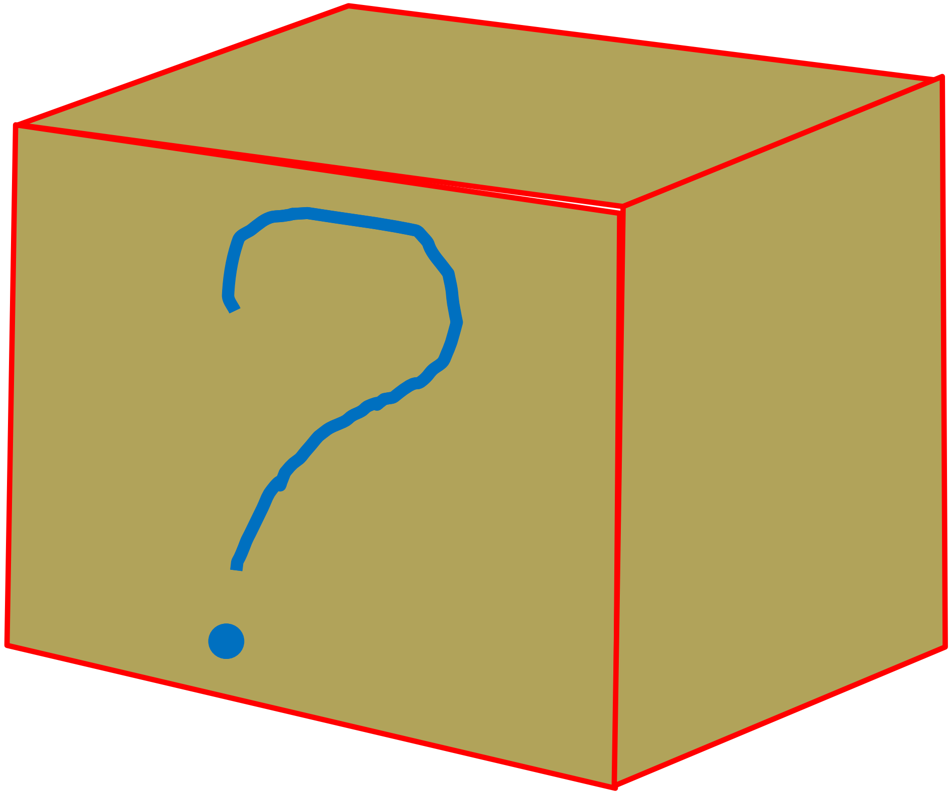


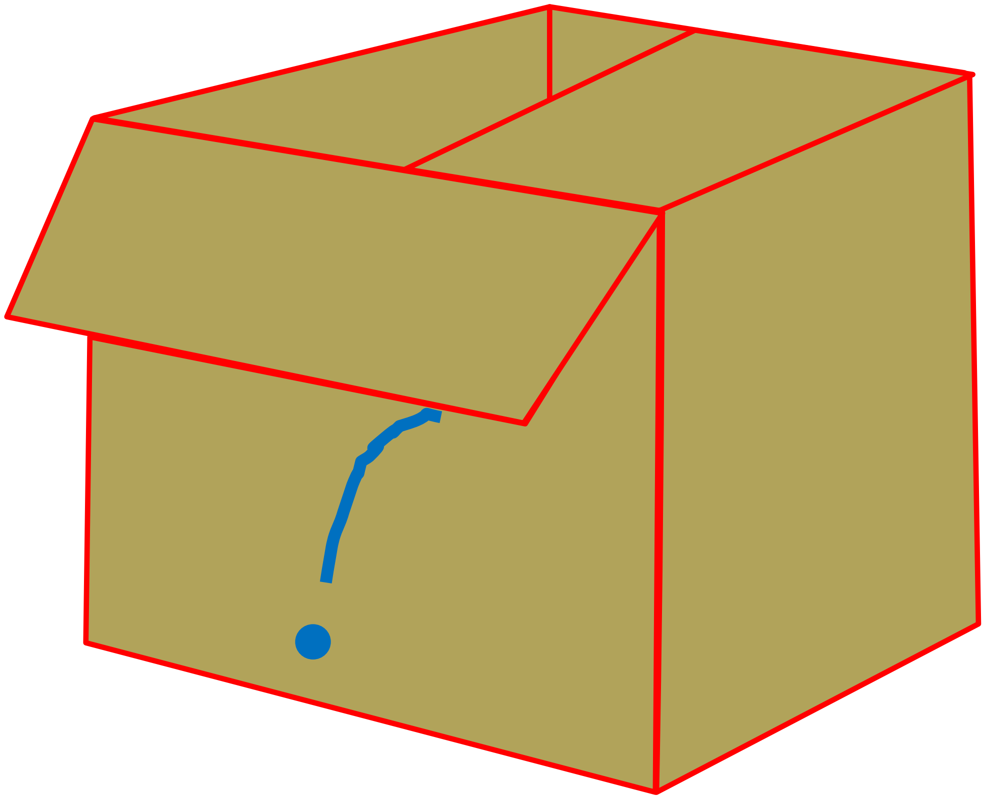
A. Einstein

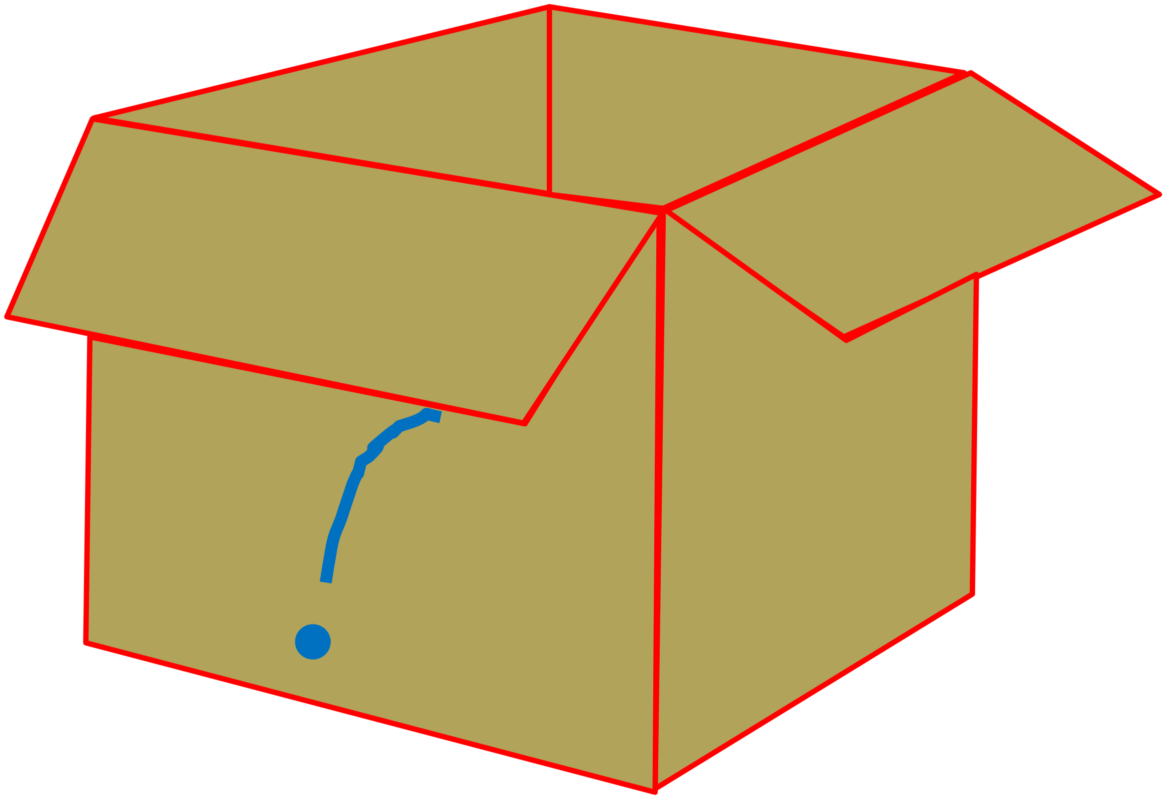
Lena

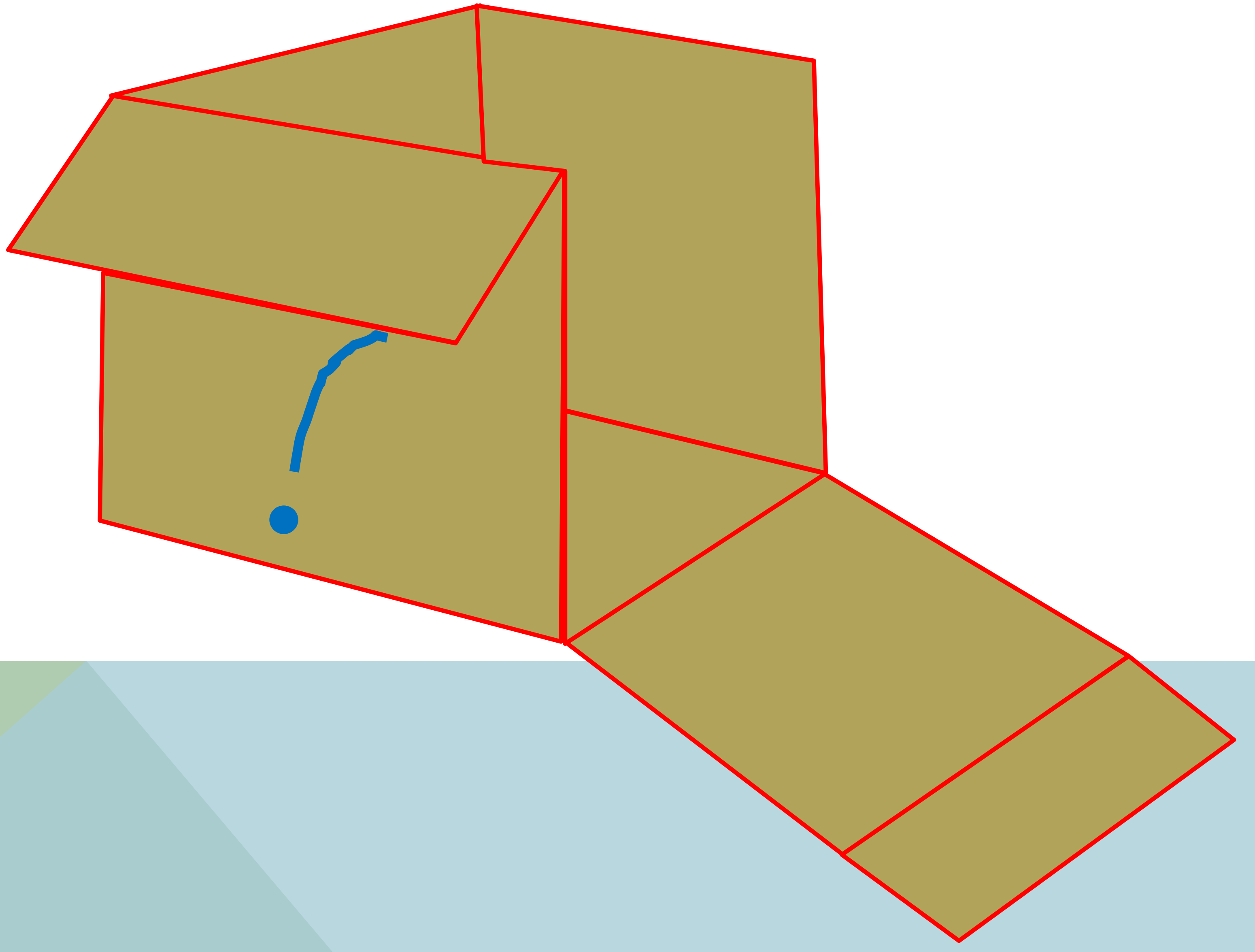


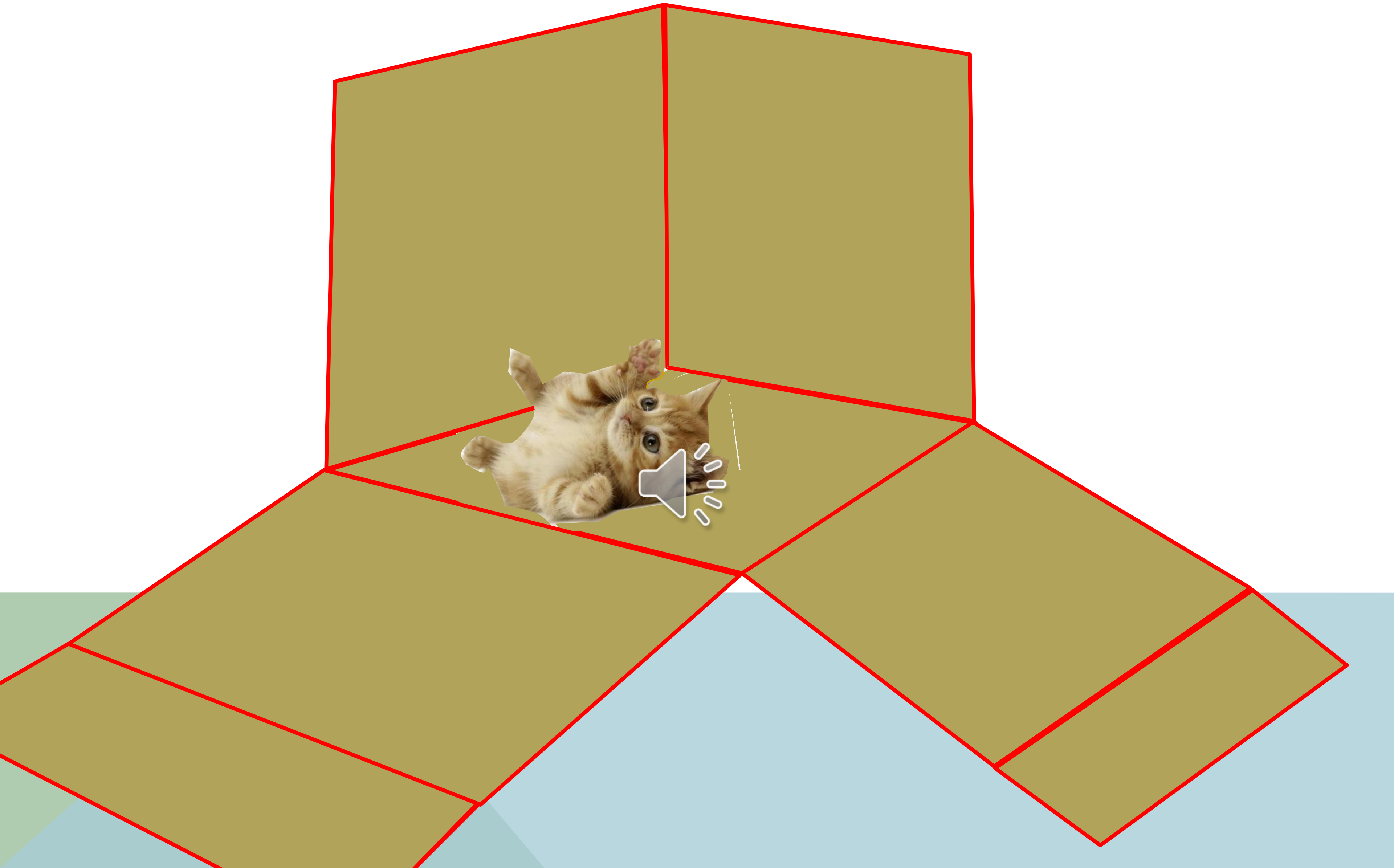
Problem: How can we know the hidden contents?



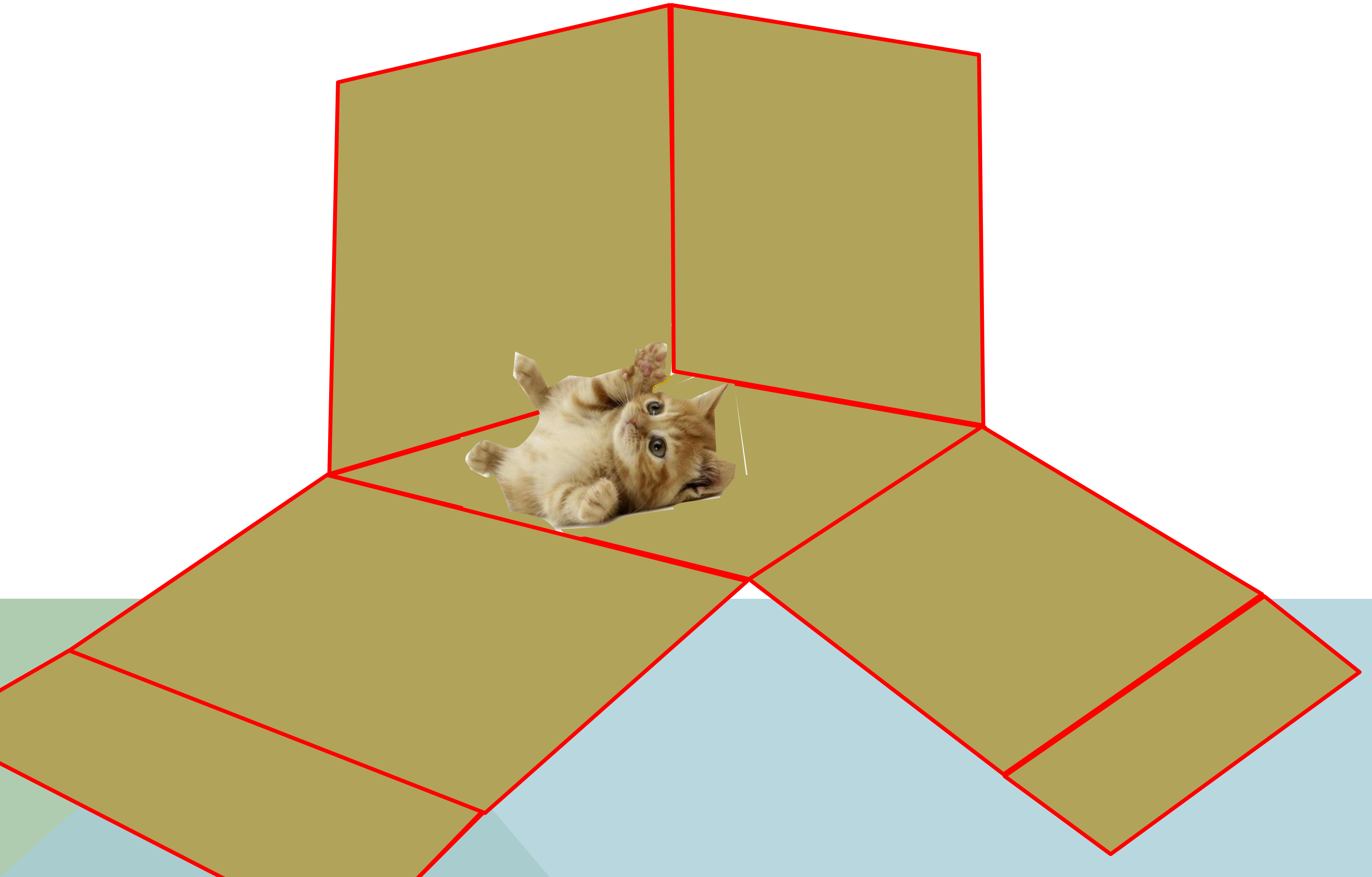








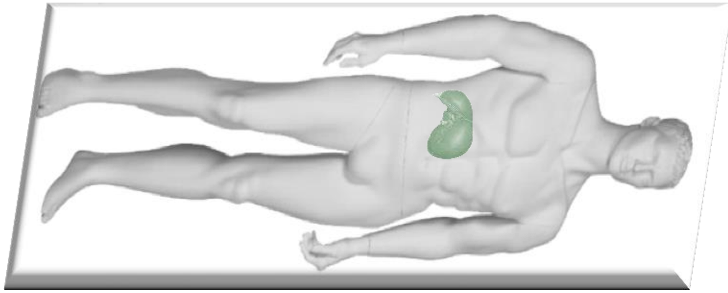
Contents known...but body destroyed

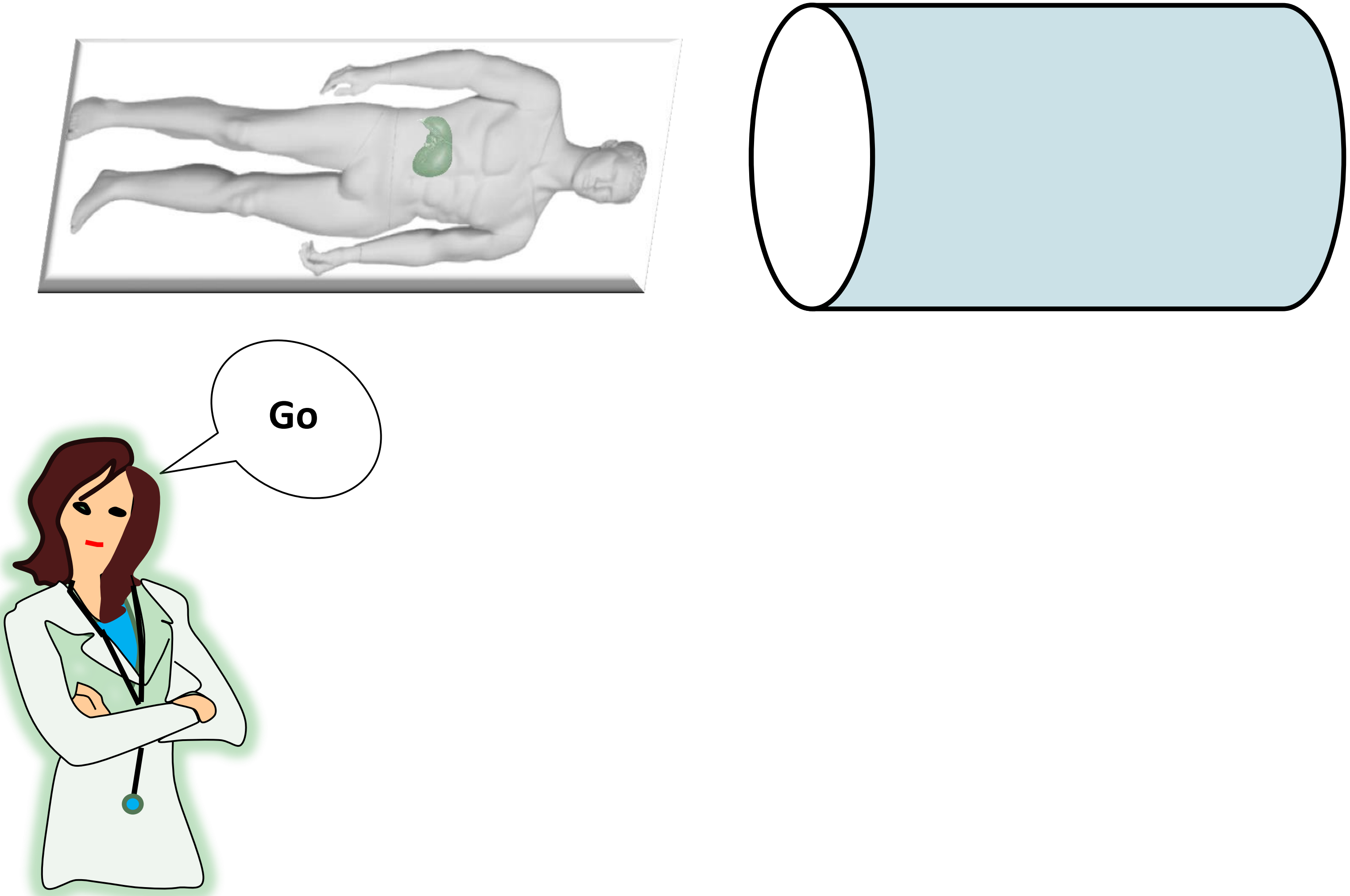


Different solution: slice the body

Different approach: slice the body

What about for the human body?

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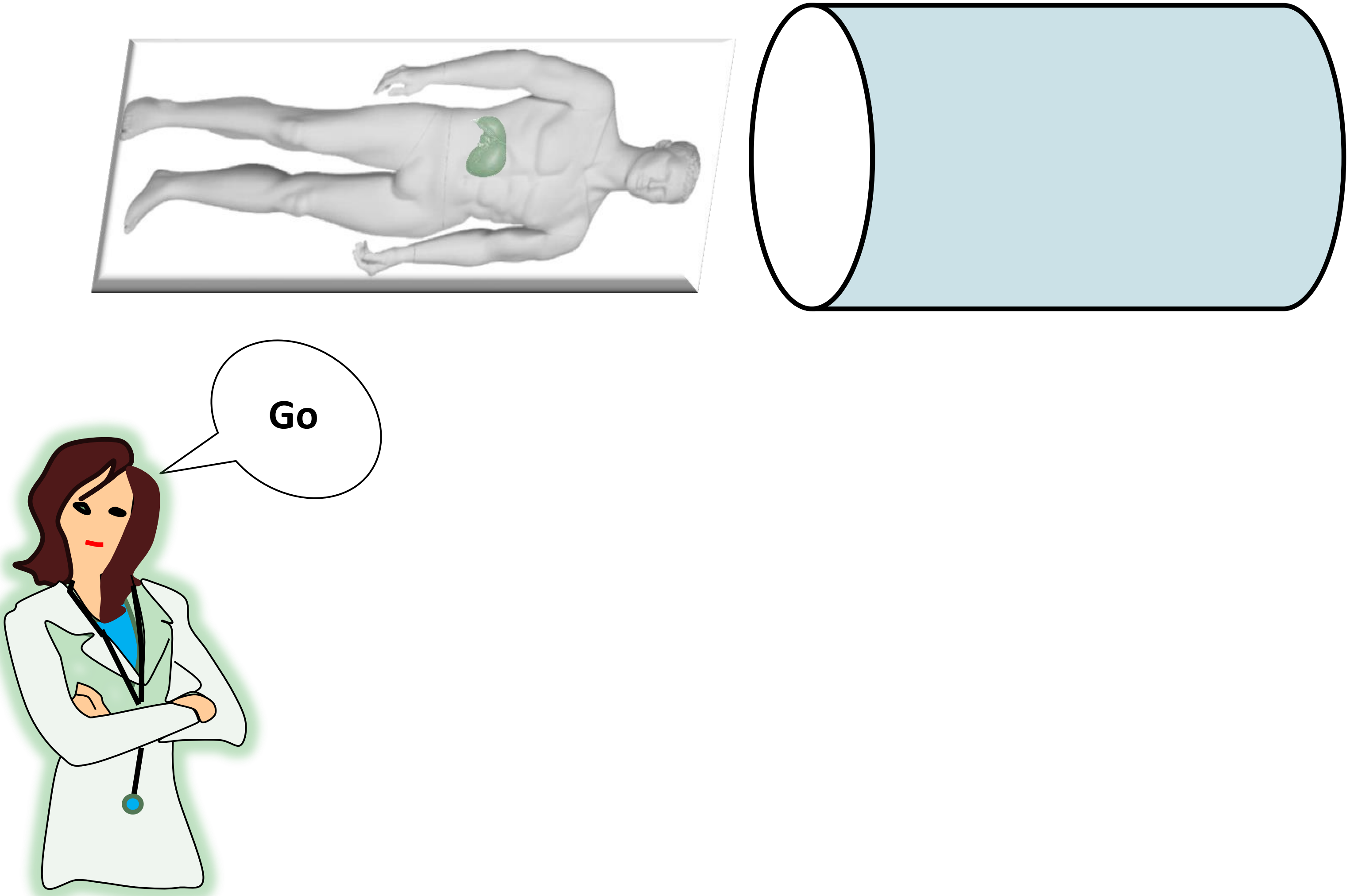
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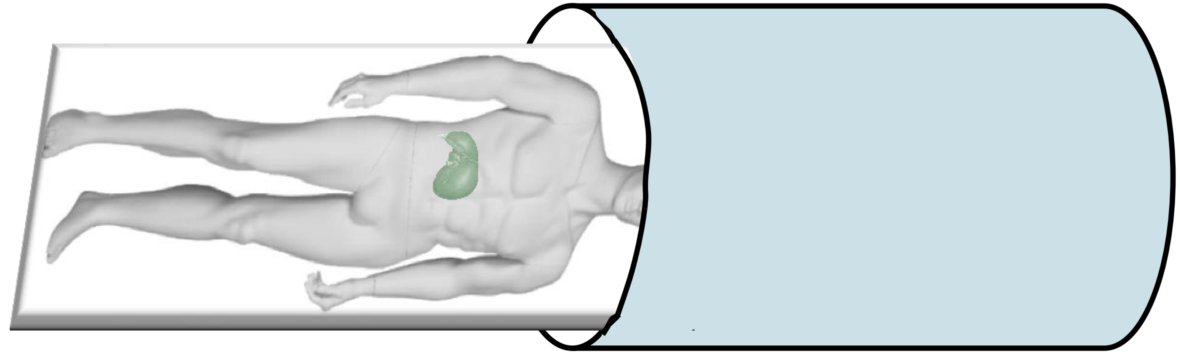


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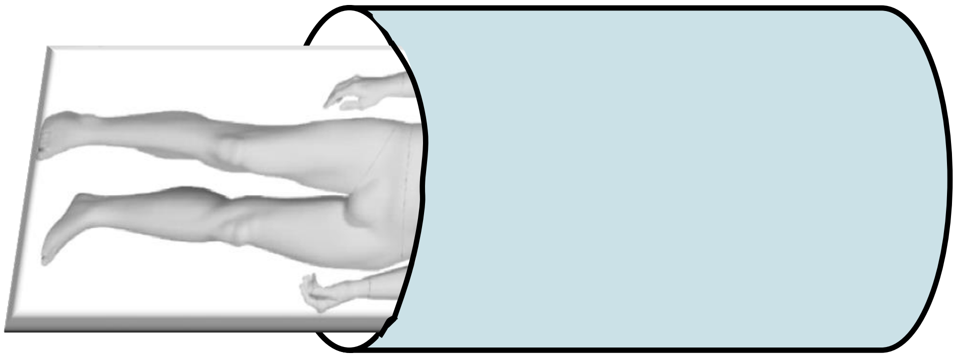


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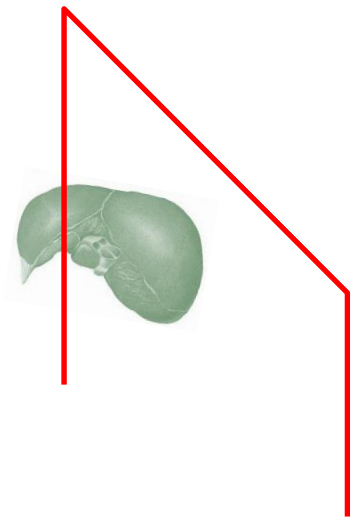

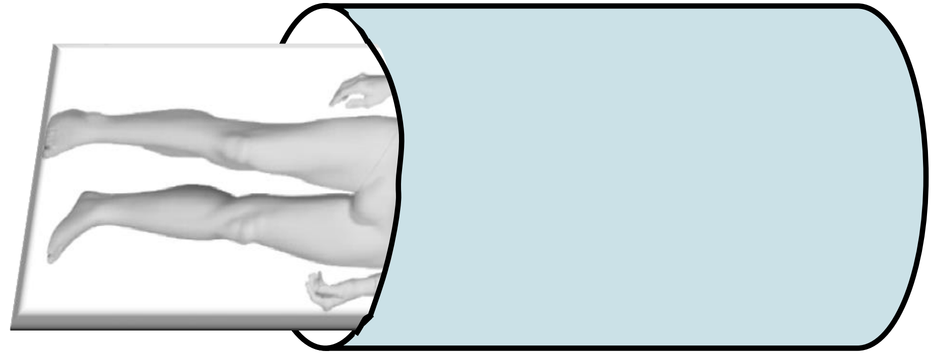


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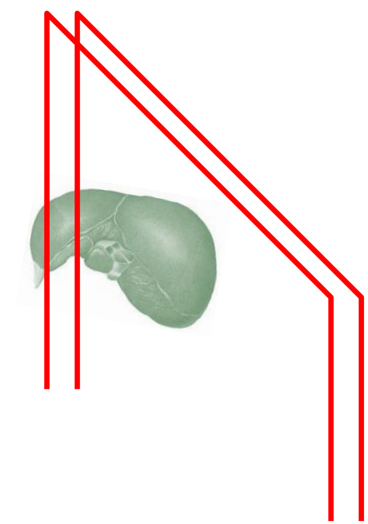

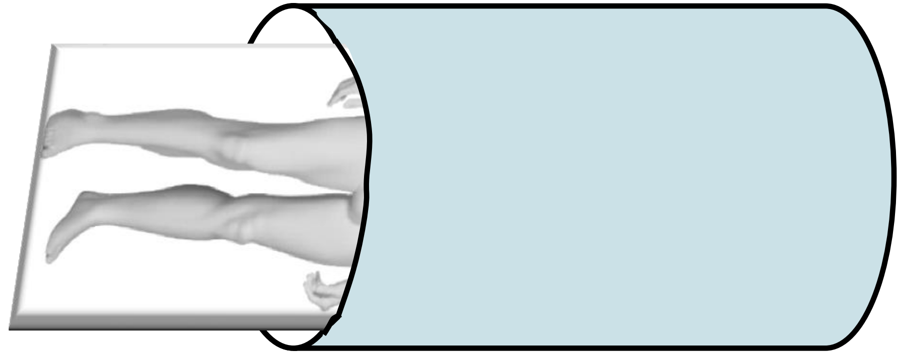


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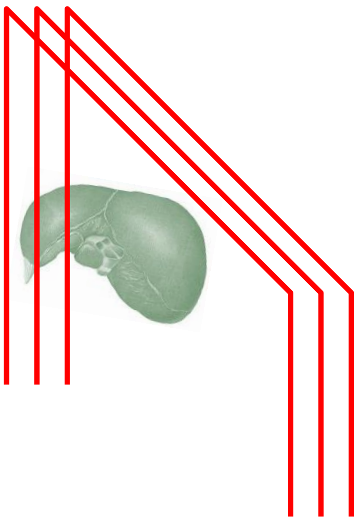

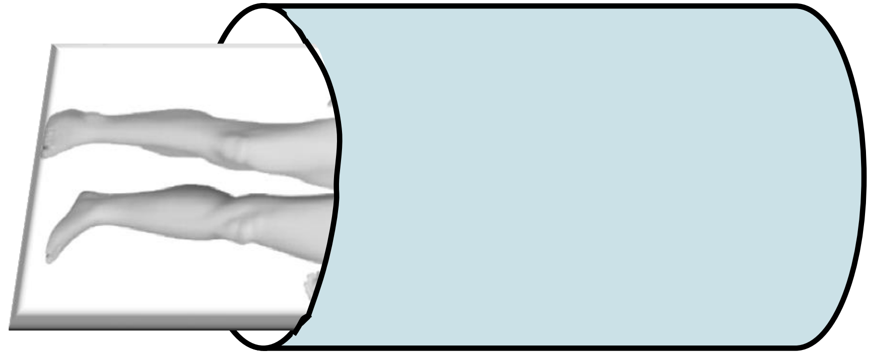


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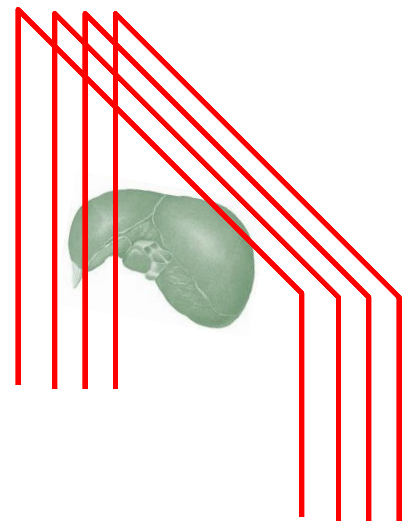


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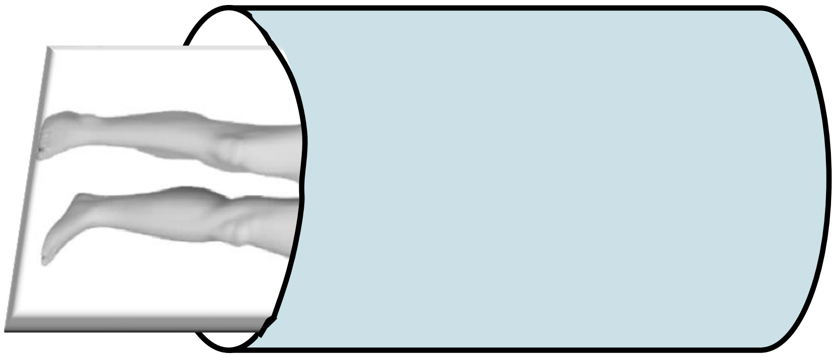

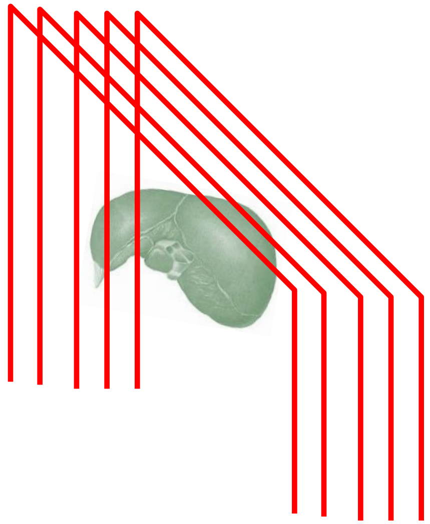

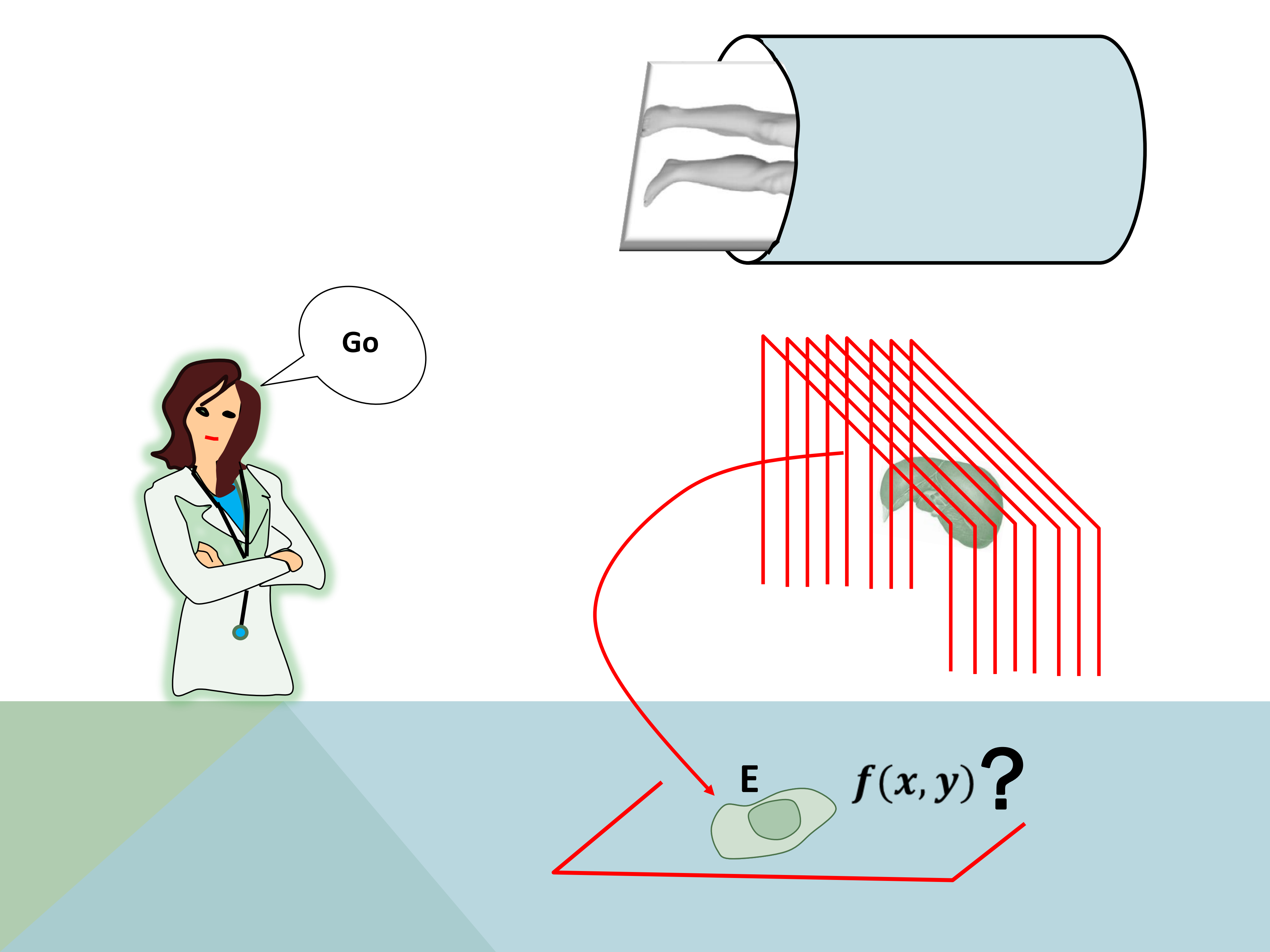
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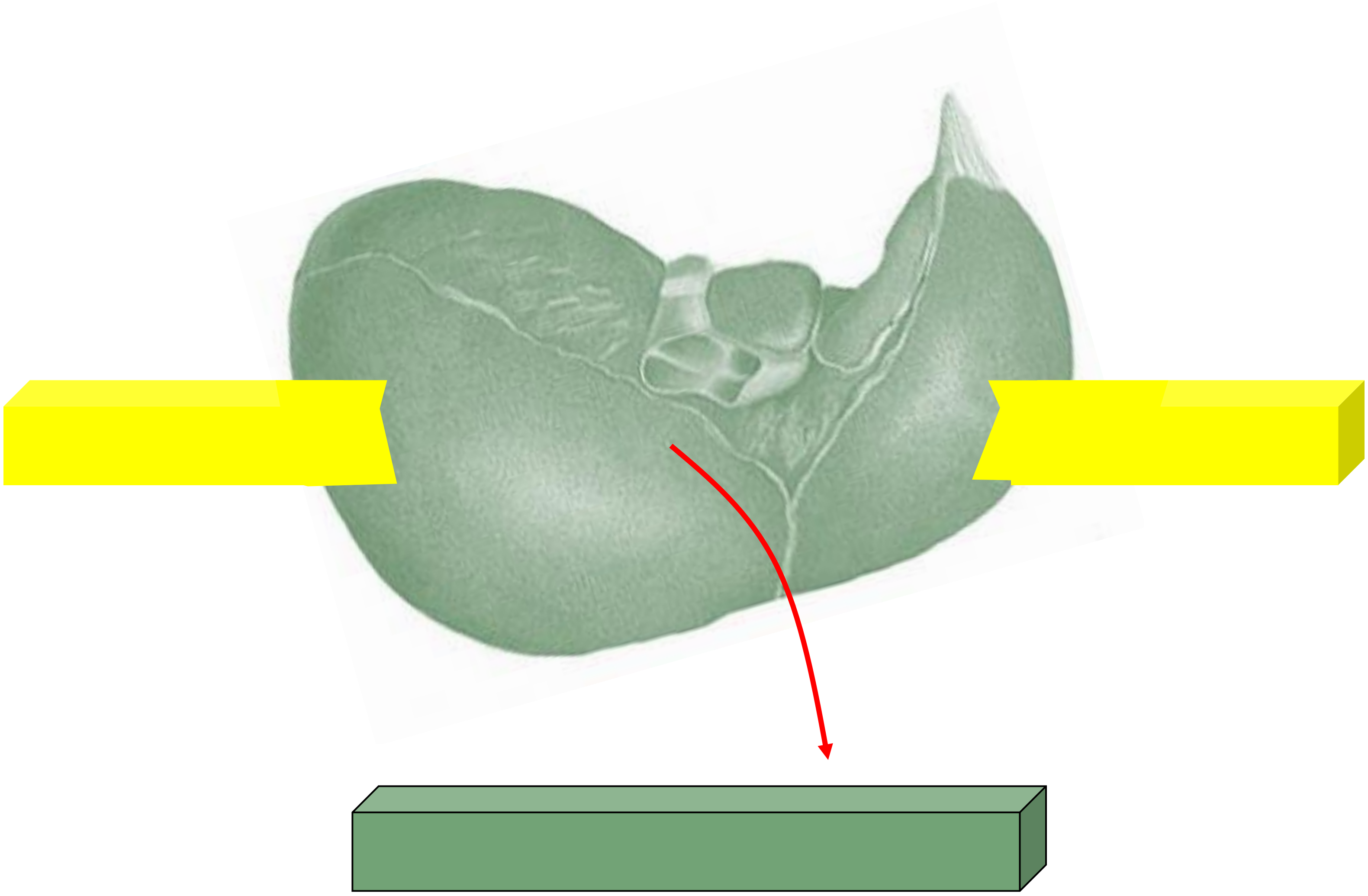
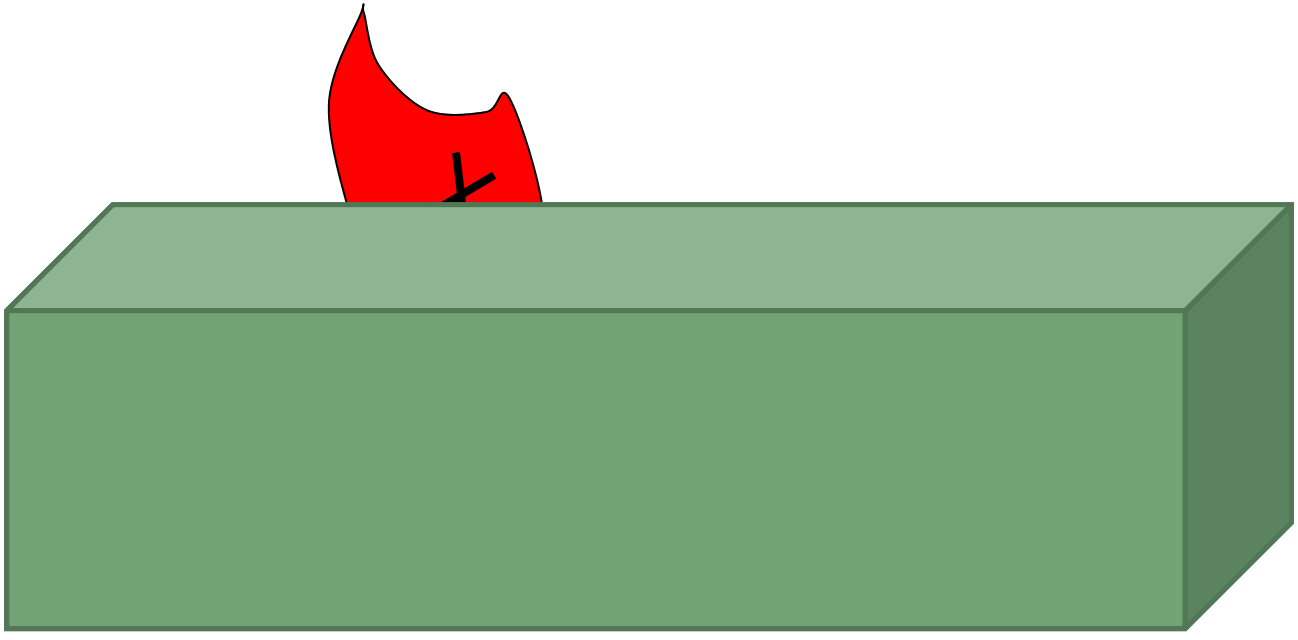
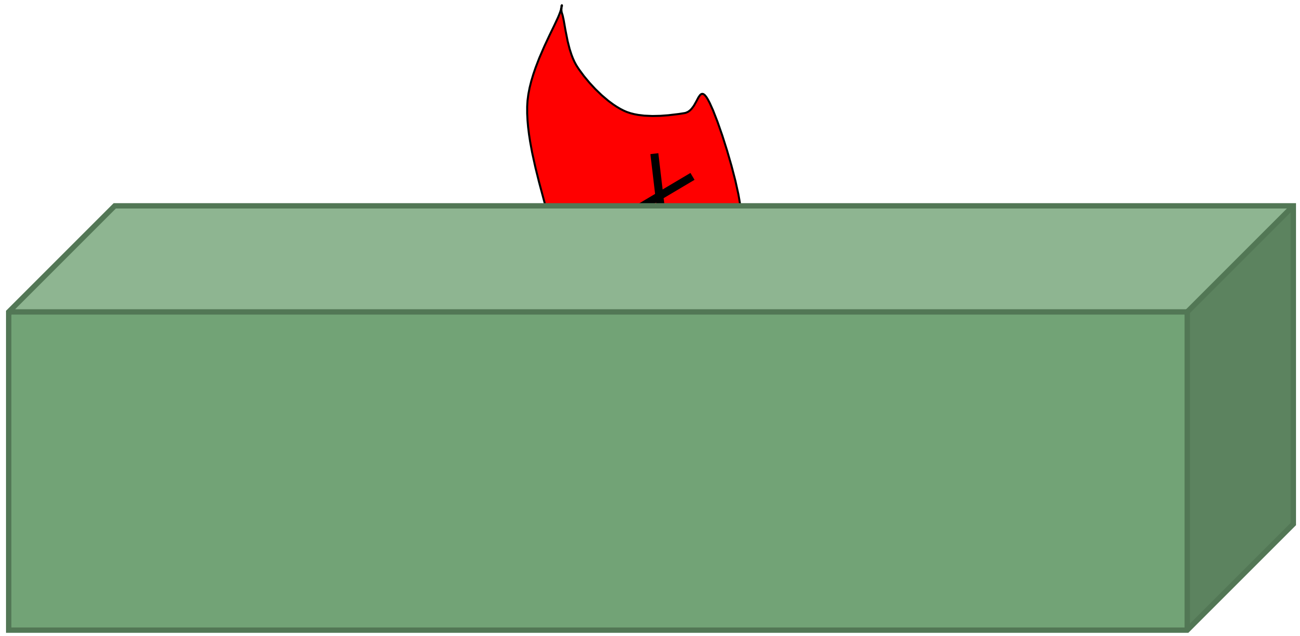
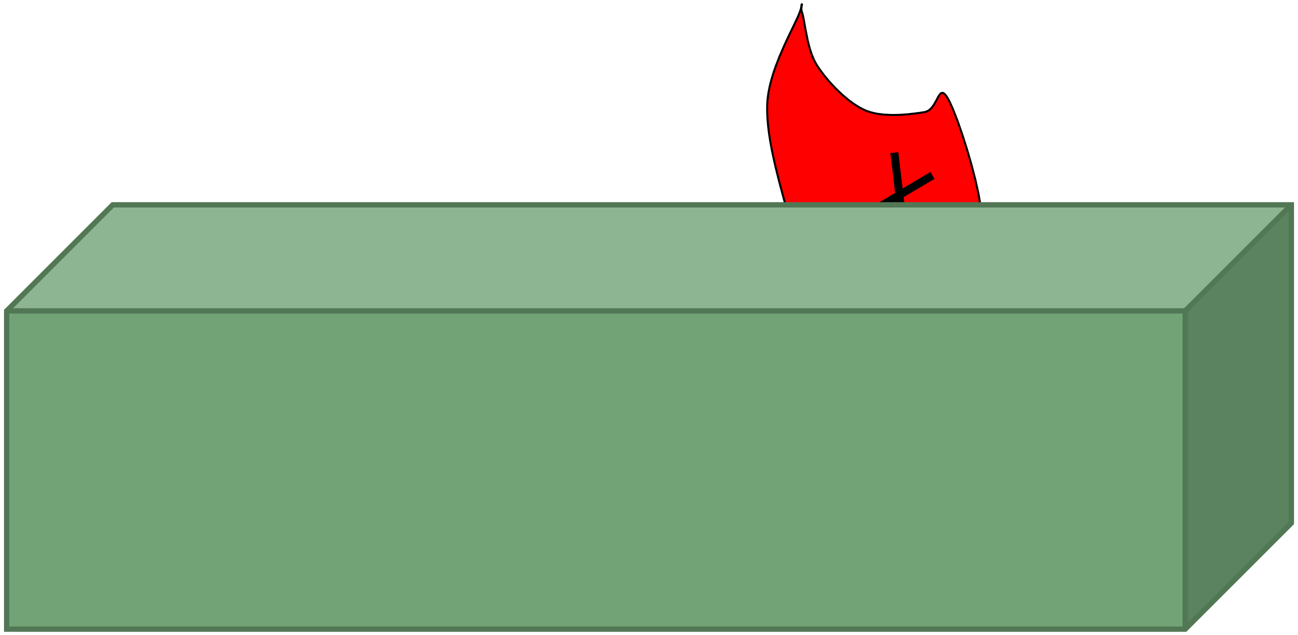
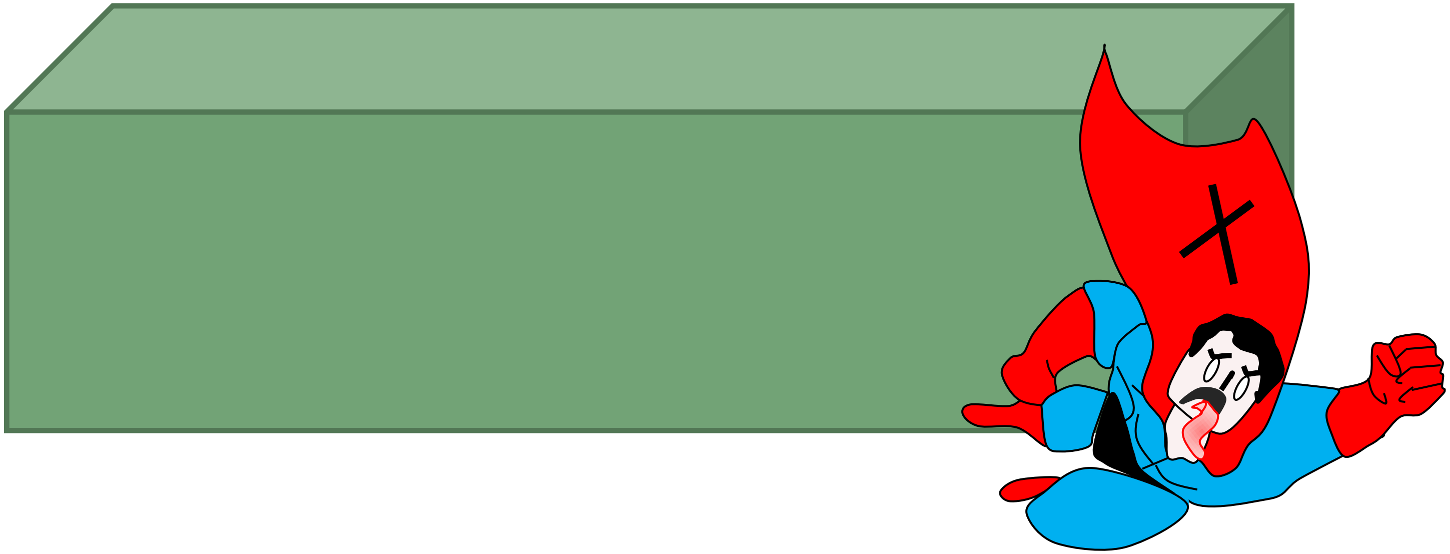
E
$f(x, y)$?

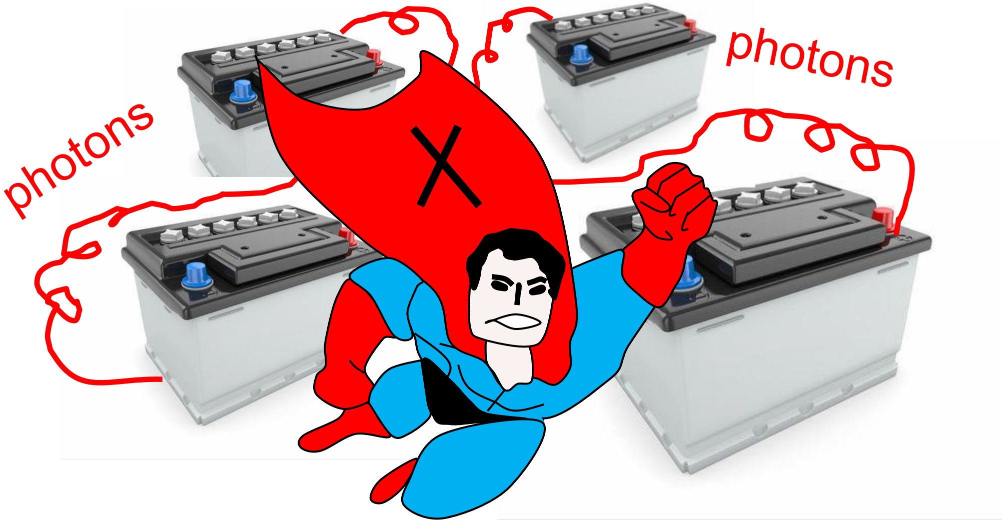


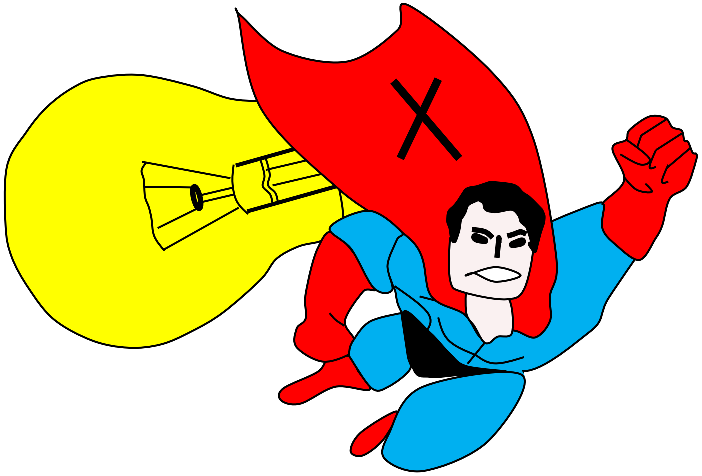


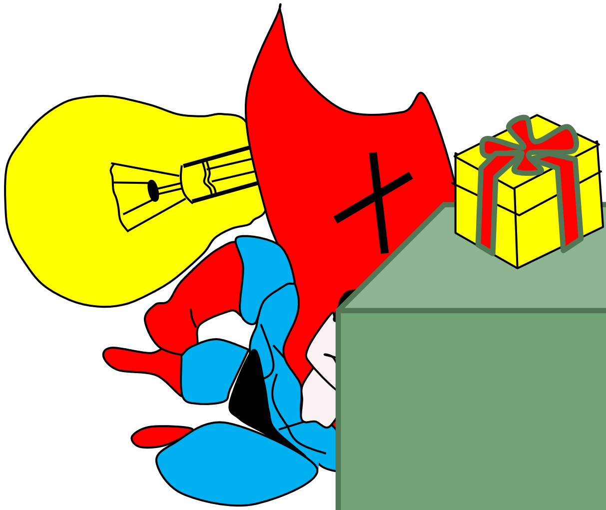


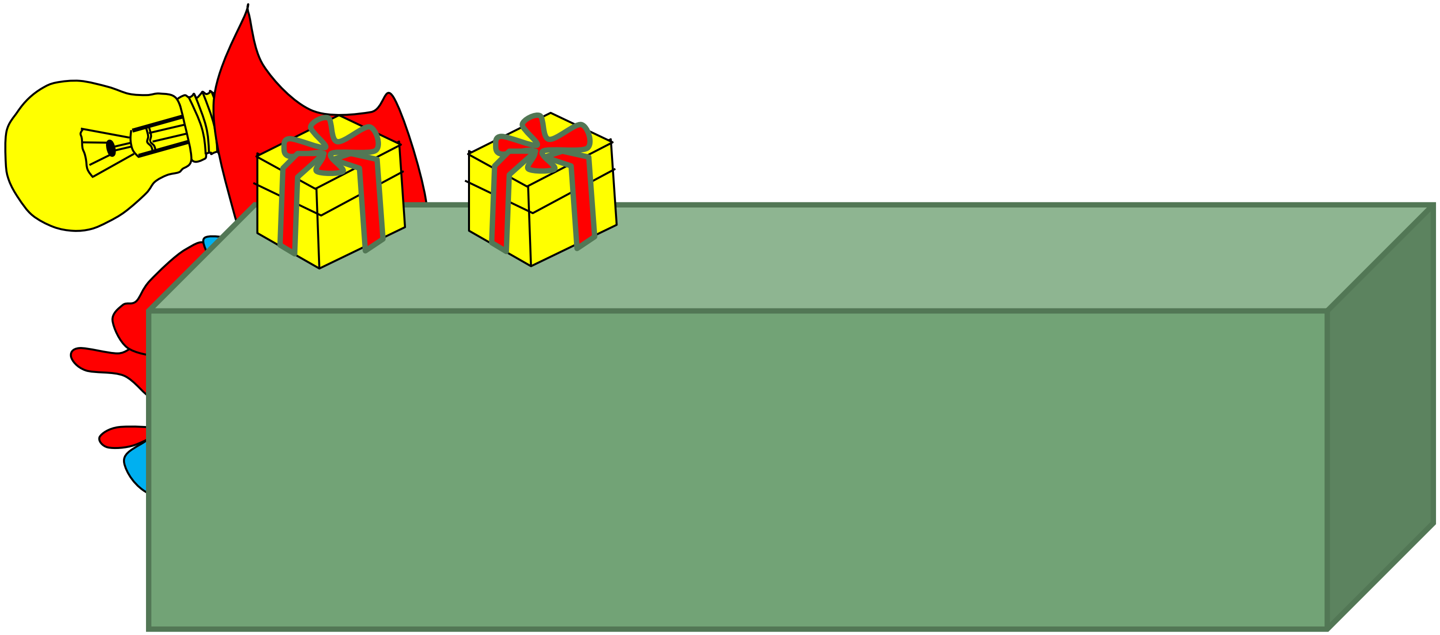


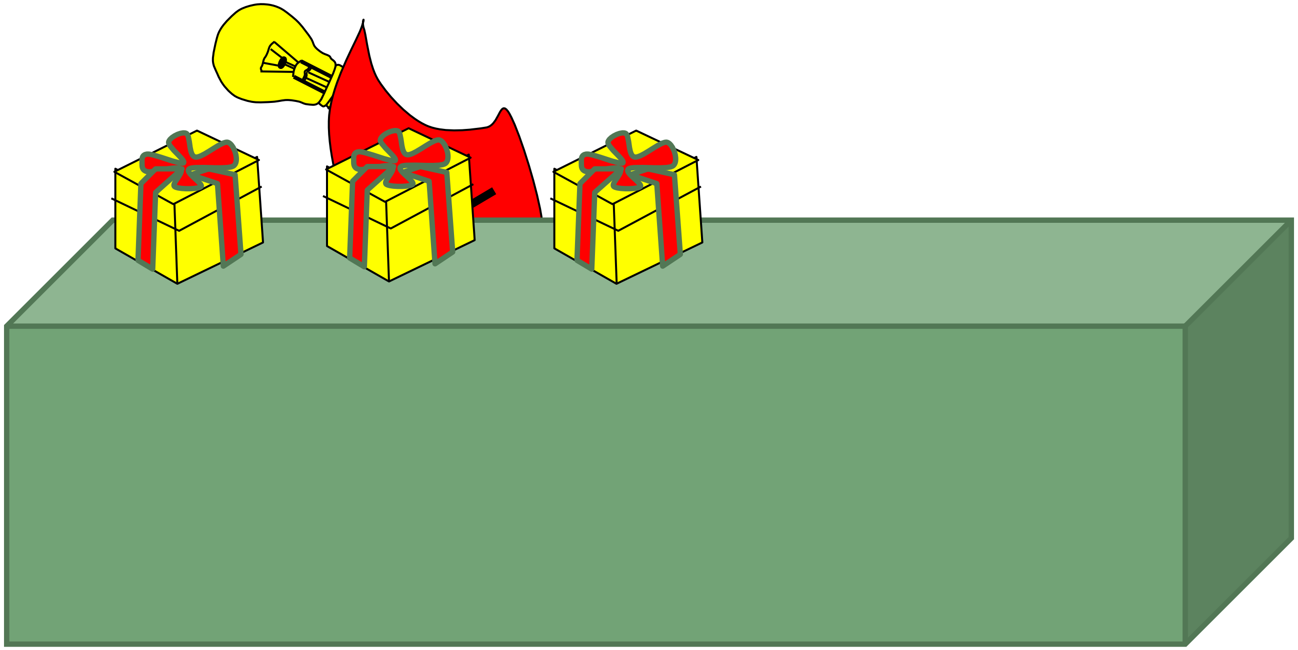


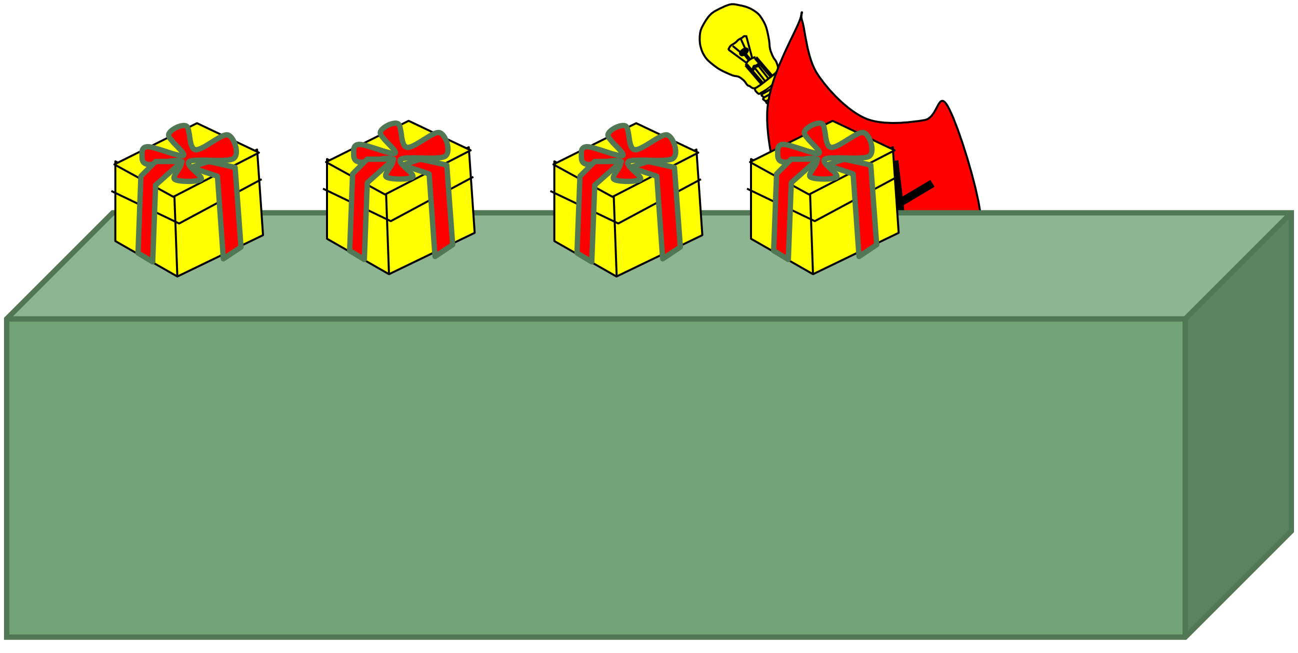

photons
photons

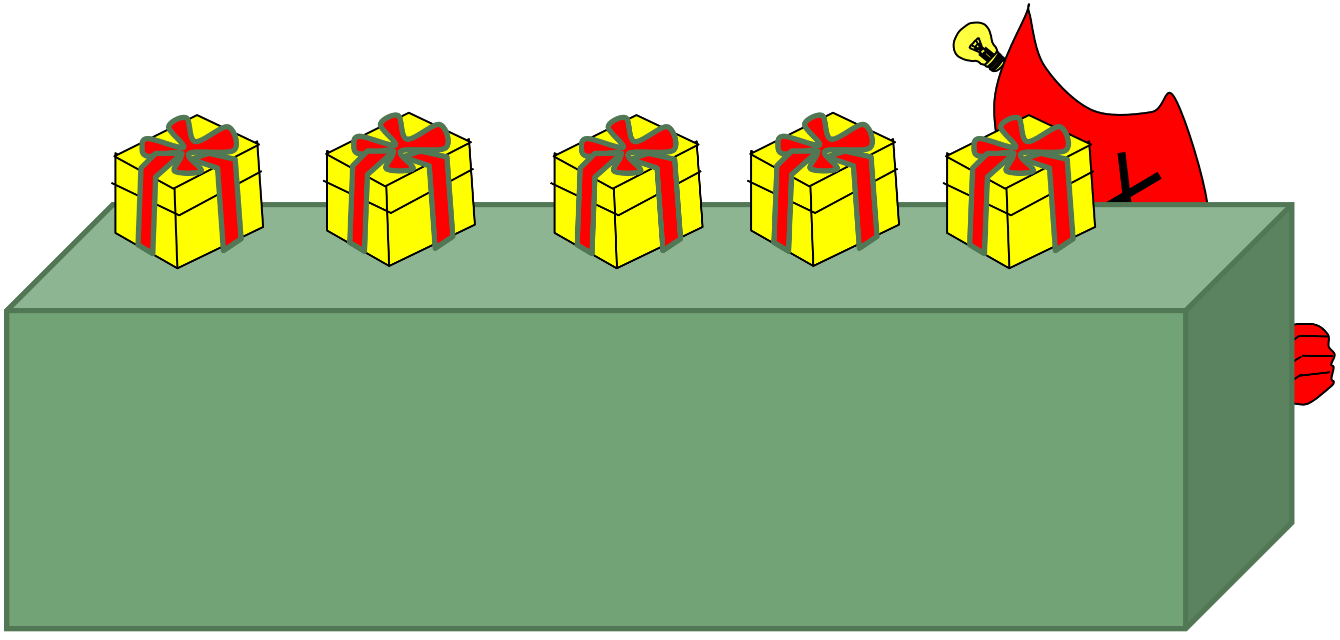


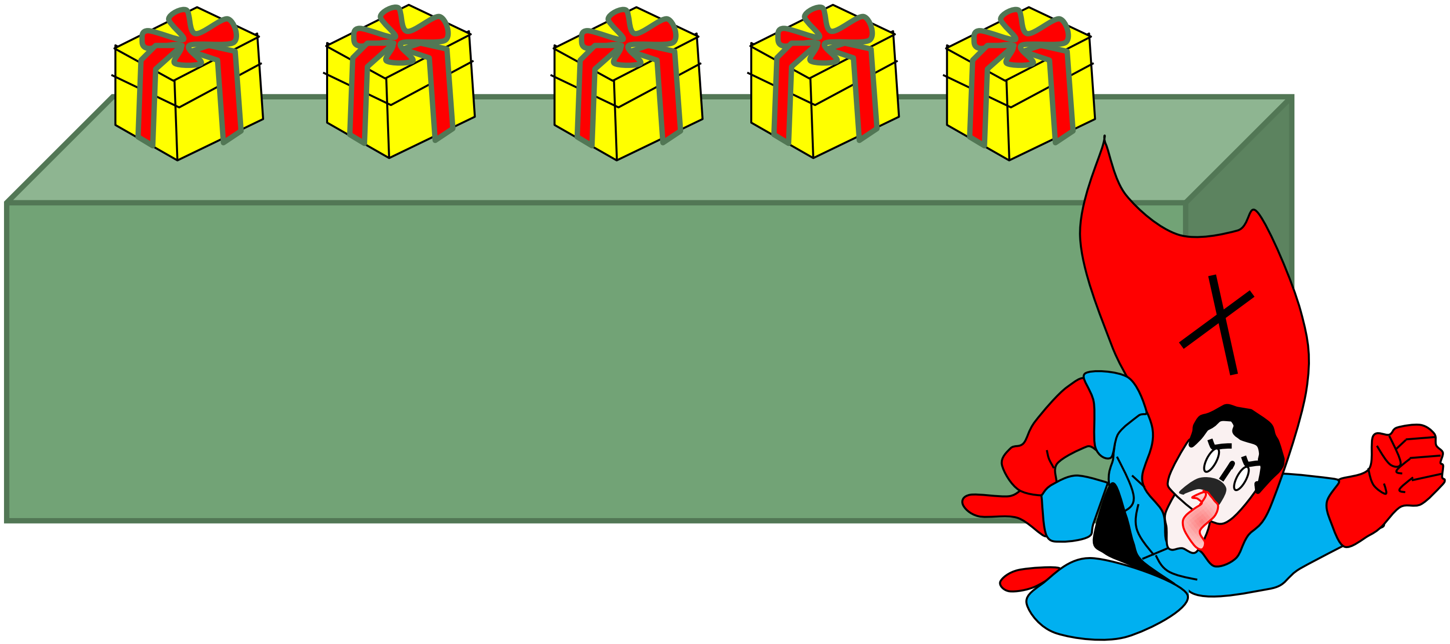






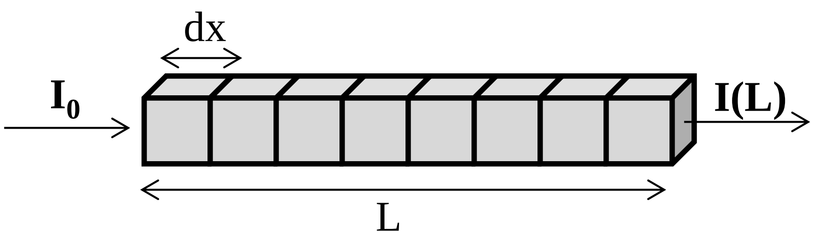






The Beer Law-1852

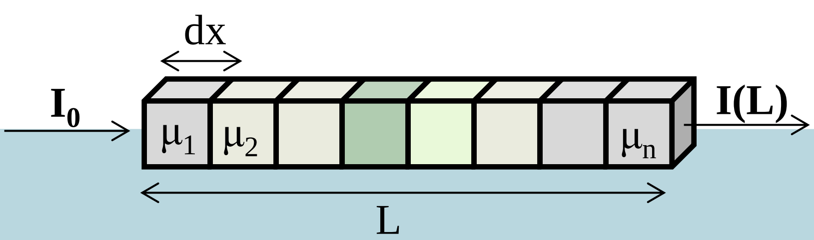
Absorbance is directly proportional to thickness



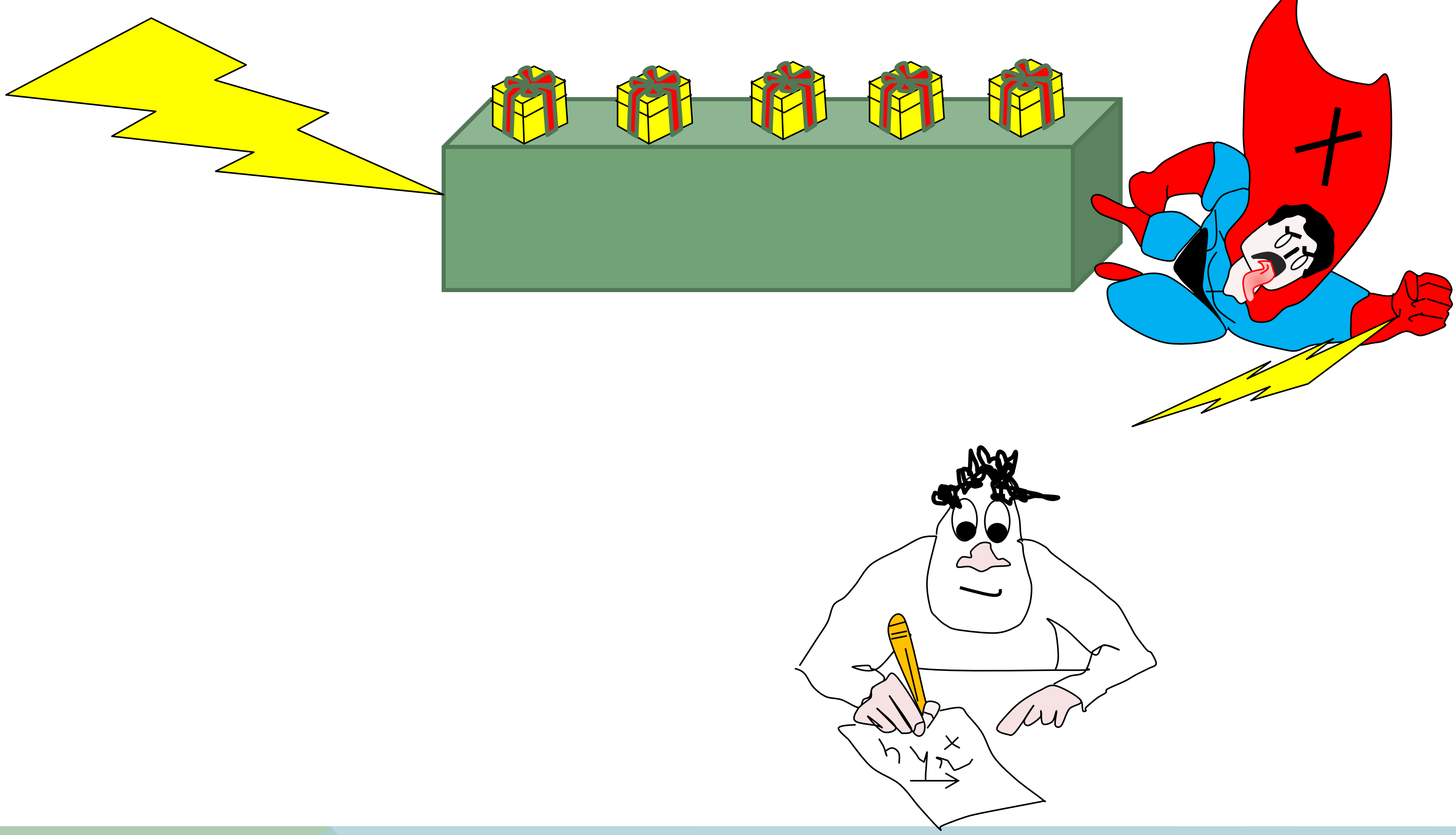
dI= intensity attenuation caused by *dx*

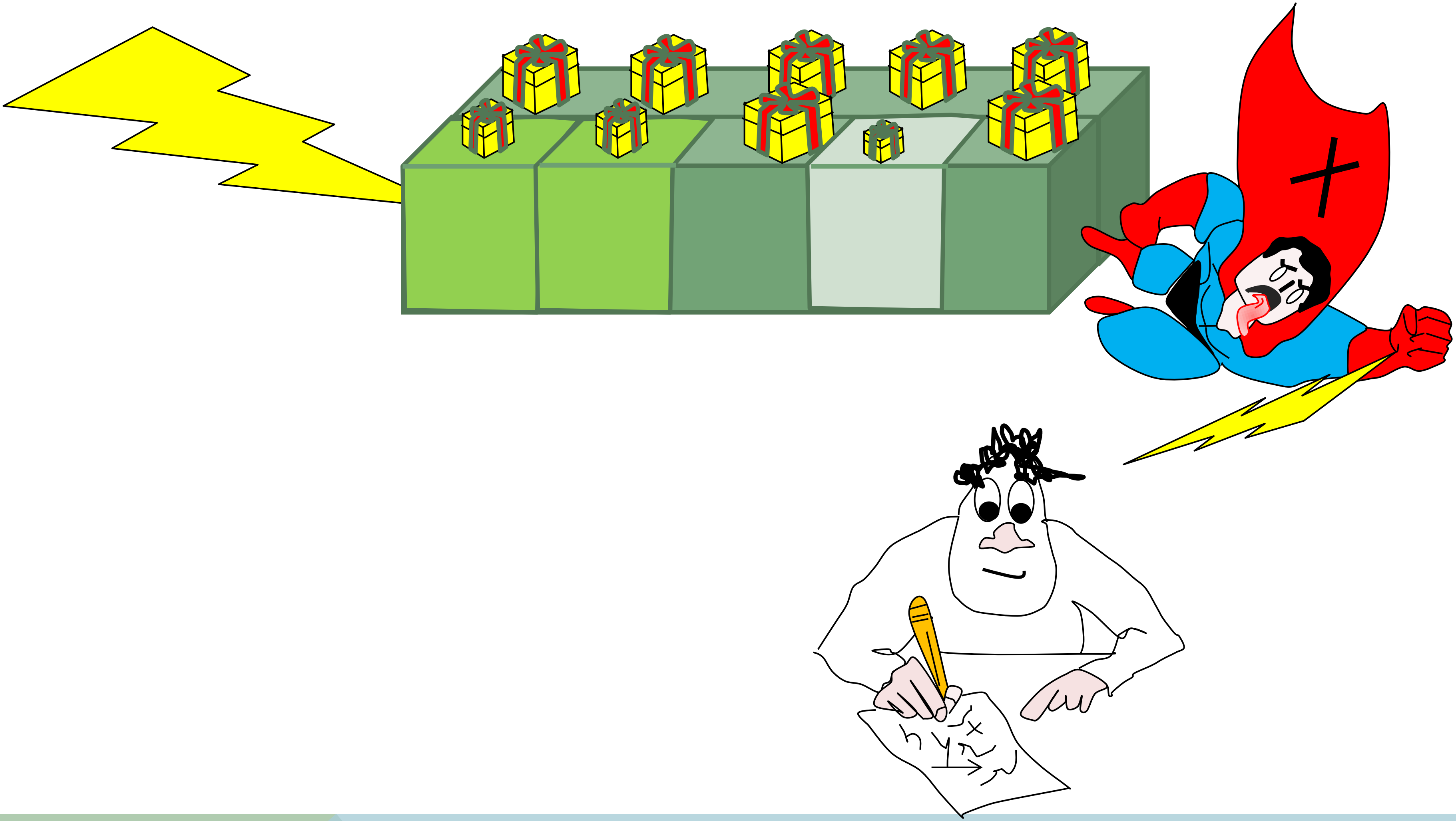
$dI = -\mu I dx$, (μ depends on the material).

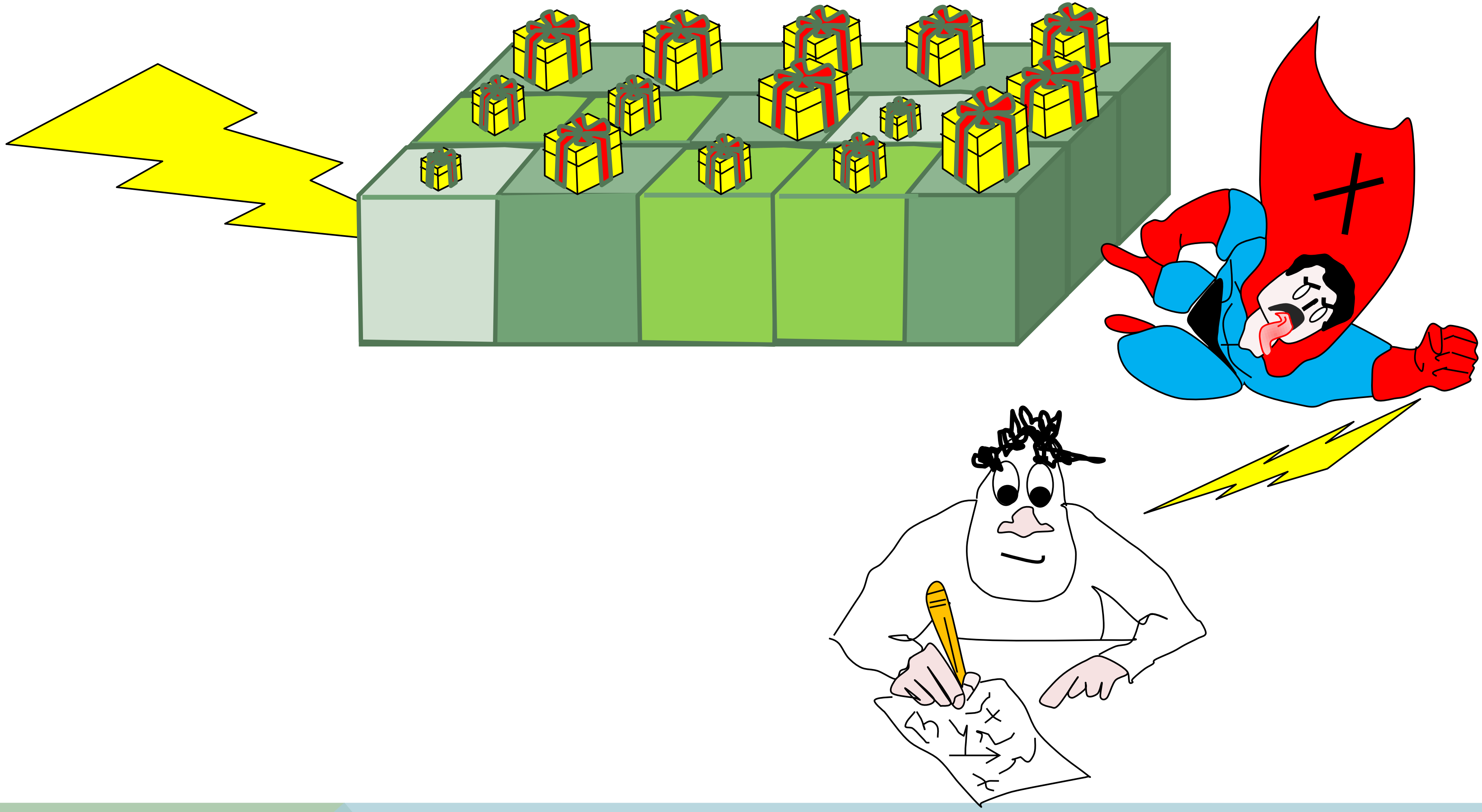
$$I(L) = I_0 e^{-\mu L}$$

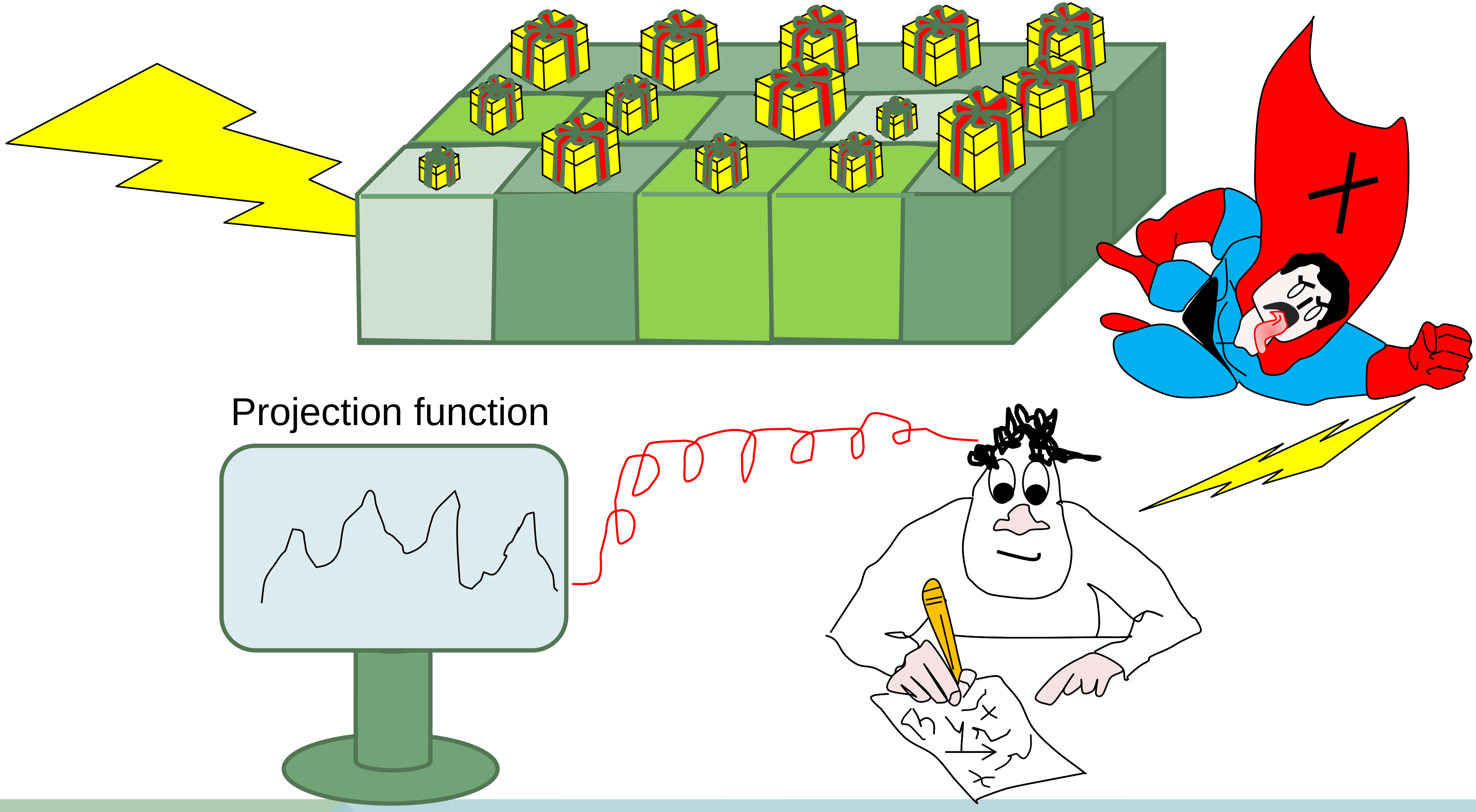


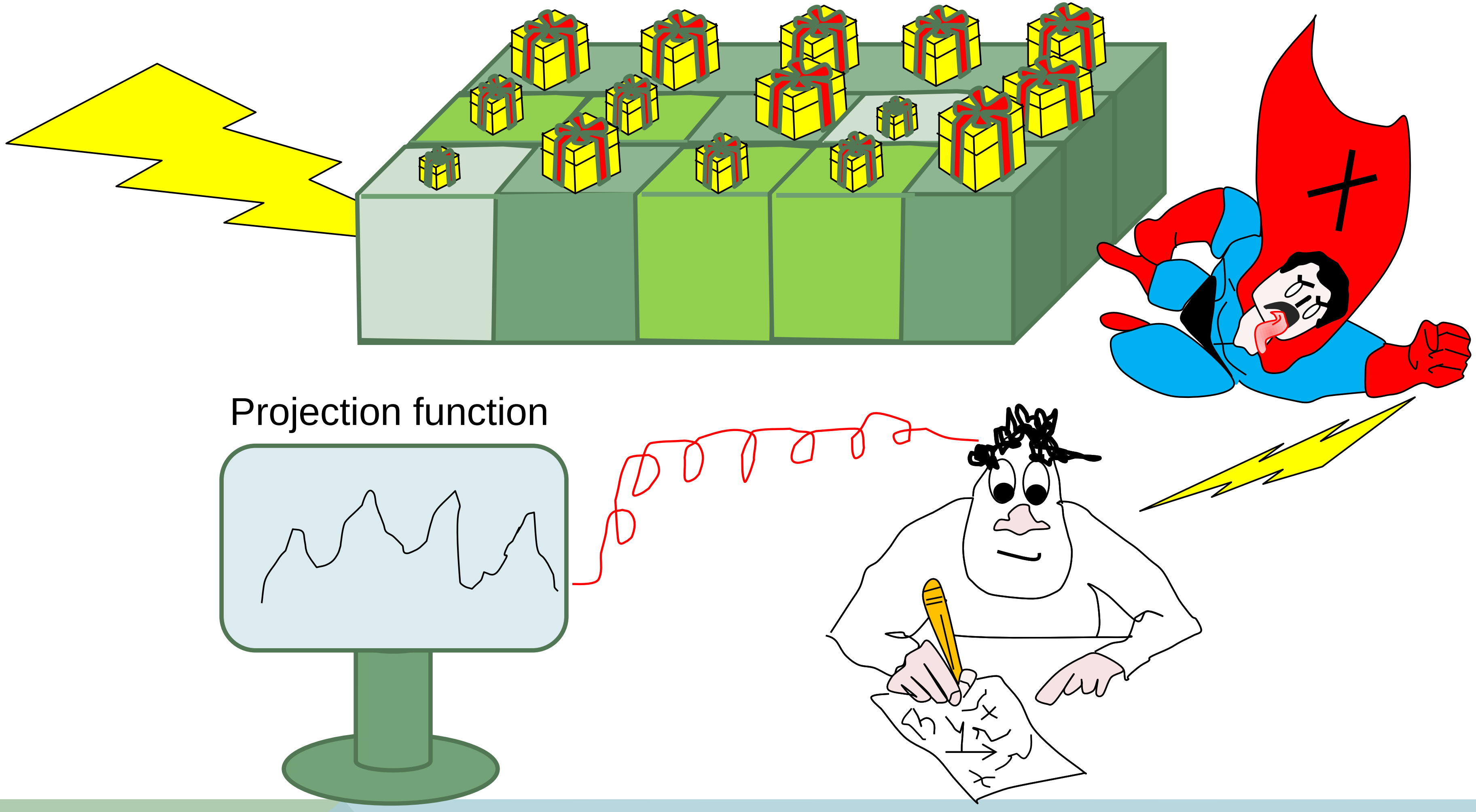
$$I(L) = I_0 e^{-(\mu 1 + \mu 2 + --- + \mu n)L}$$

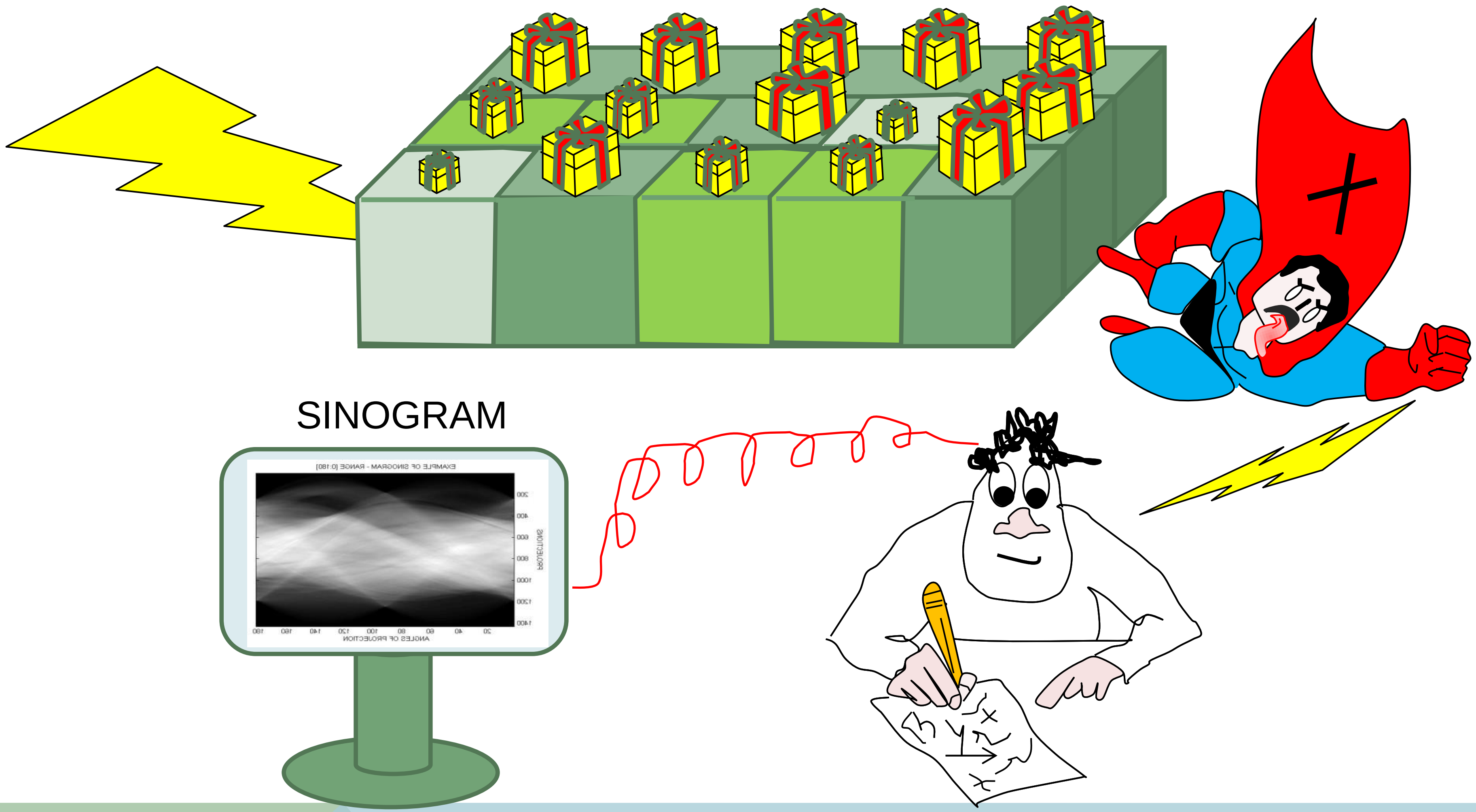


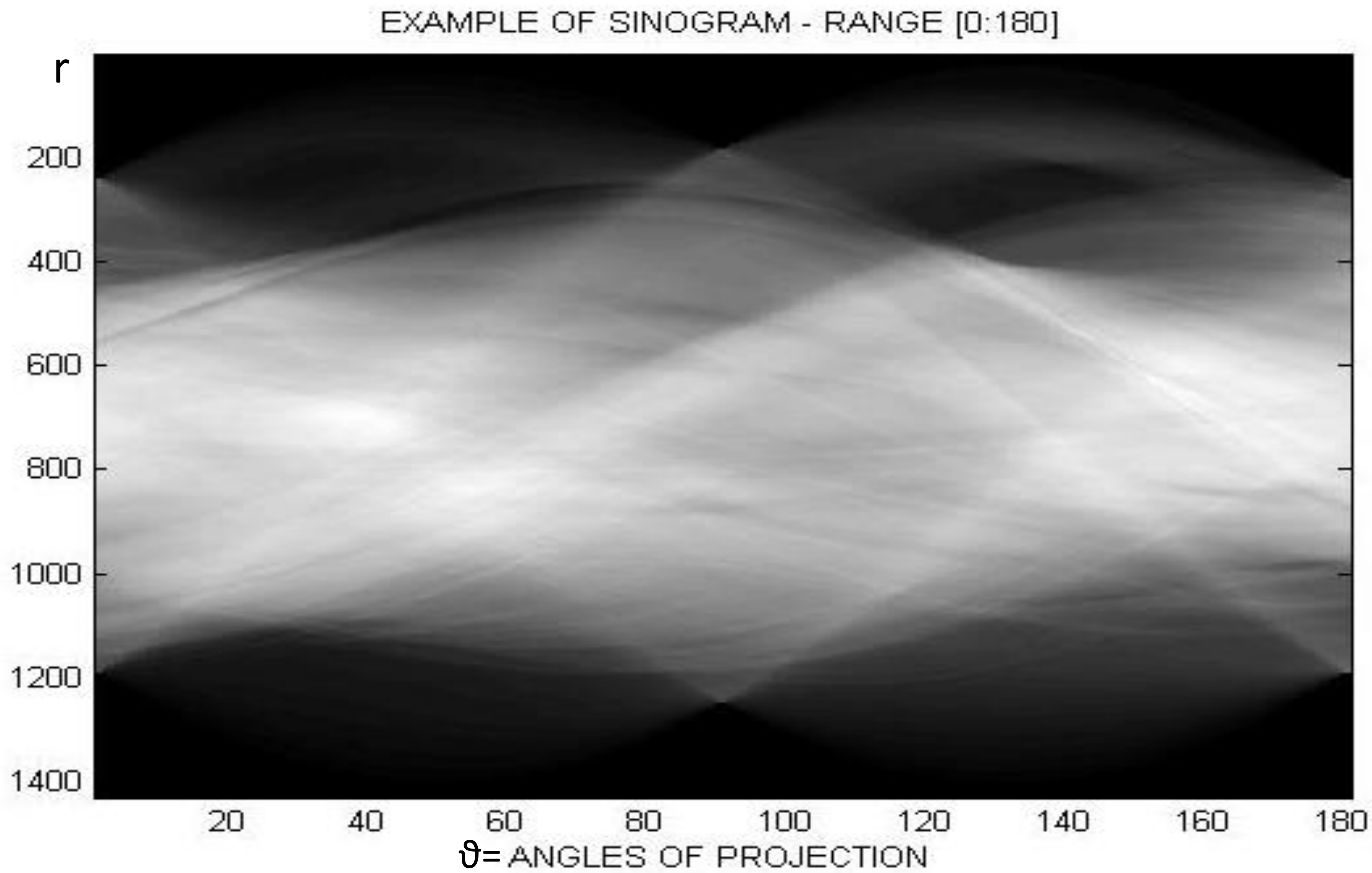


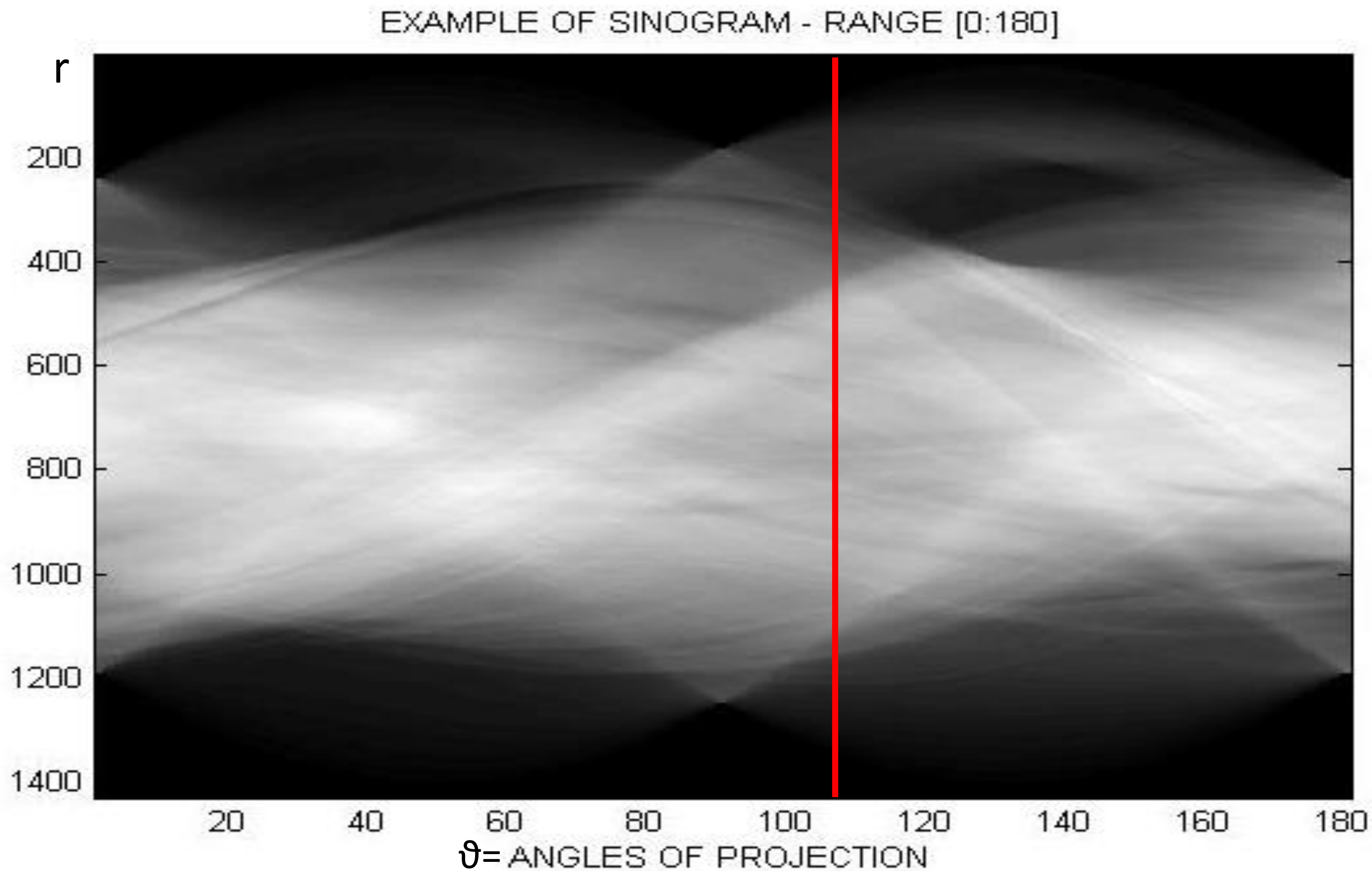



Projection function


Projection function


SINOGRAM


EXAMPLE OF SINOGRAM - RANGE [0:180]
r
200
400
600
800
1000
1200
1400
20
40
60
80
100
120
140
160
180
ϑ= ANGLES OF PROJECTION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r
200
400
600
800
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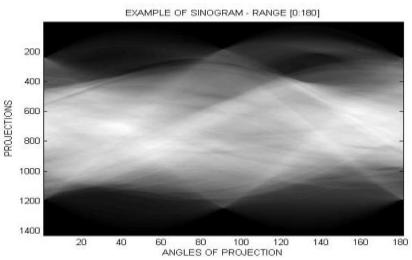
120
140
160
180
ϑ= ANGLES OF PROJECTION

RADON TRANSFORM

The sinogram provides the visualization of the available data set

Rf

Radon Transform of the density function f(x,y)



$$\{p(r,\vartheta),\ \ \vartheta \in [0,\pi)\}$$

Johann Radon (1887-1956)

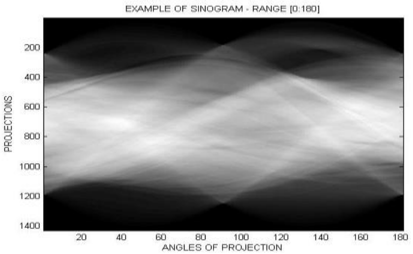
RADON TRANSFORM

The sinogram provides the visualization of the available data set

Rf

Radon Transform of the density function f(x,y)

||



||

$$\{p(r,\vartheta),\ \ \vartheta \in [0,\pi)\}$$

Johann Radon (1887-1956)

MAIN PROBLEM: INVERSION OF THE RADON TRANSFORM

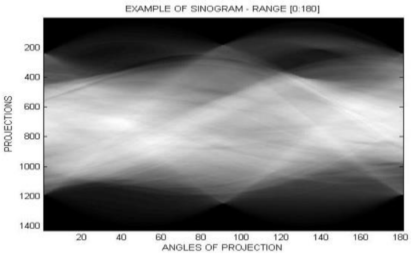
RADON TRANSFORM

The sinogram provides the visualization of the available data set

$\xrightarrow{Rf}$

Radon Transform of the density function f(x,y)

||

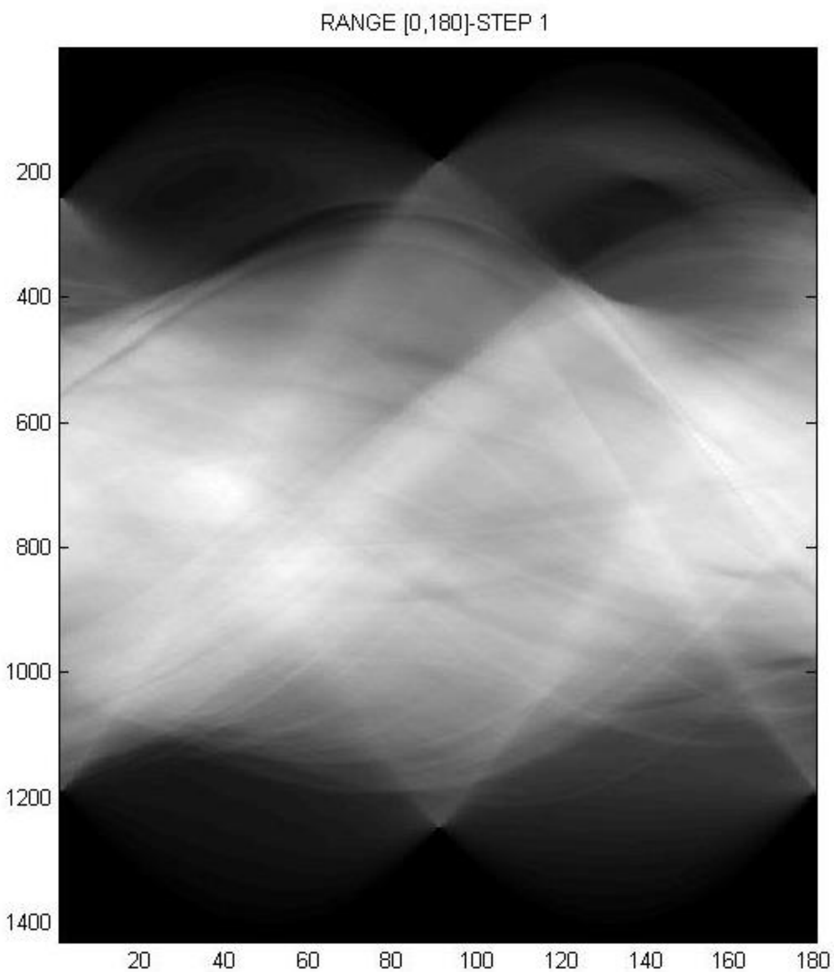


||

$$\{p(r,\vartheta),\ \ \vartheta \in [0,\pi)\}$$

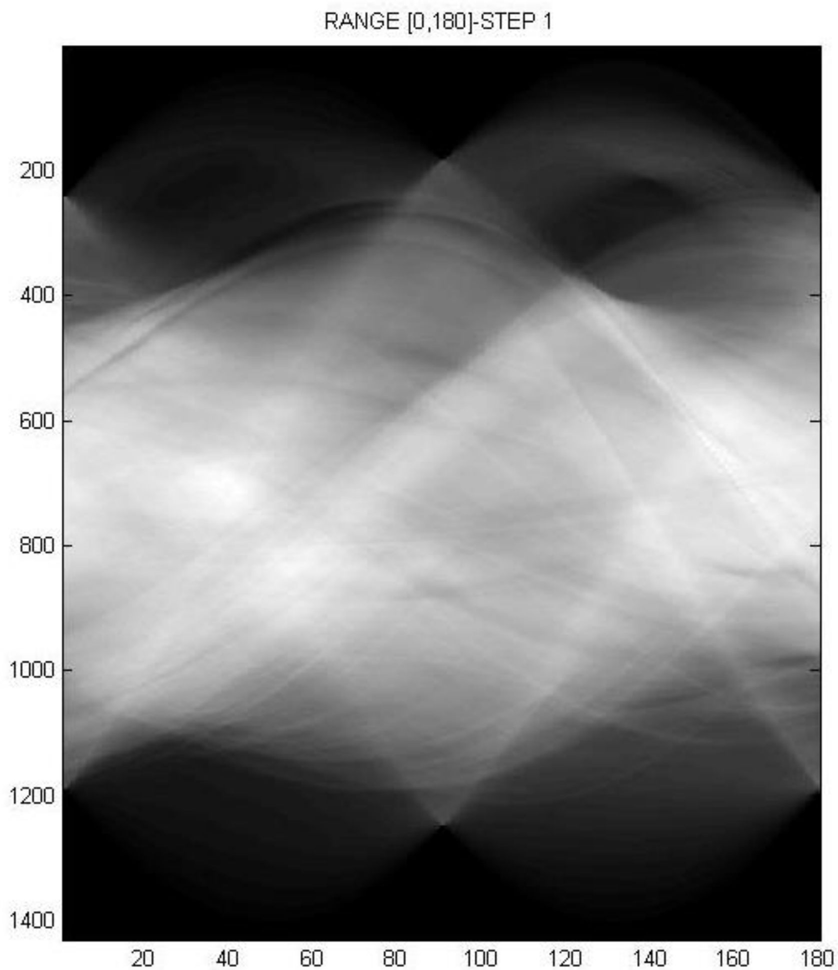
Johann Radon (1887-1956)

J. Radon, "**Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten**", Berichte über die Verhandlungen der Königlich-Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse, Leipzig: Teubner (69): 262–277,1917


RANGE [0,180]-STEP 1
200
400
600
800
1000
1200
1400
20
40
60
80
100
120
140
160
180

INGREDIENTS

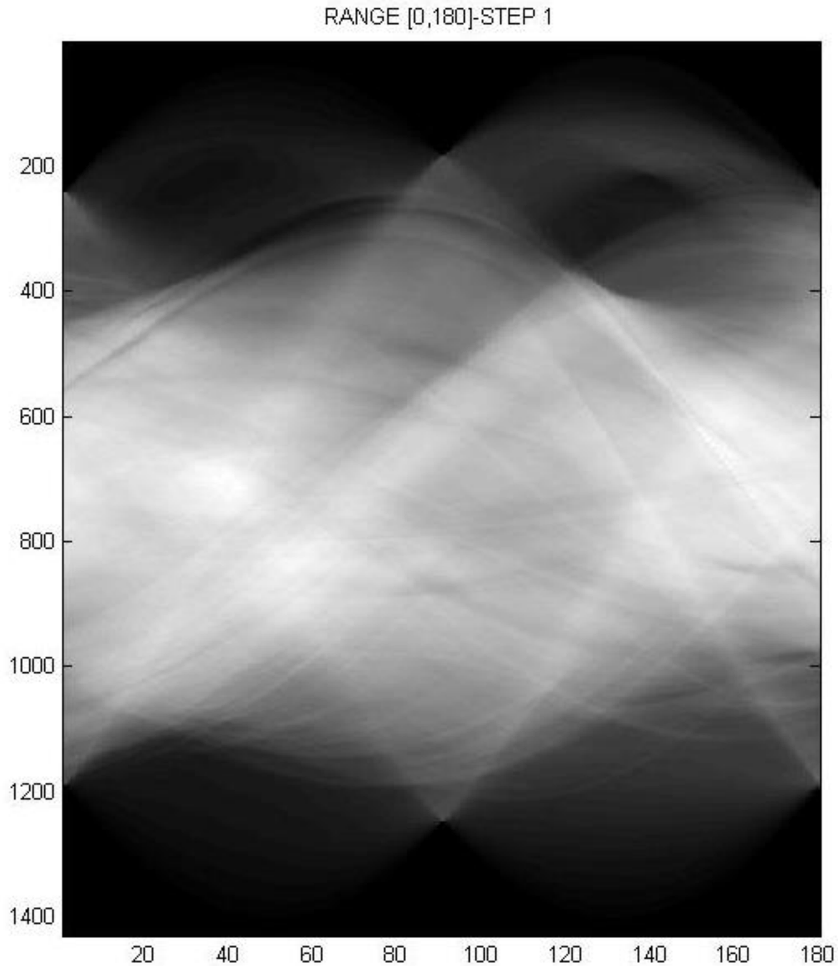
Radon Transform
Fourier Transform
Riesz operator
Back-projection
Filtering



INGREDIENTS

Radon Transform
Fourier Transform
Riesz operator
Back-projection
Filtering

$$f = \frac{1}{4\pi} B I^{-1} R f$$

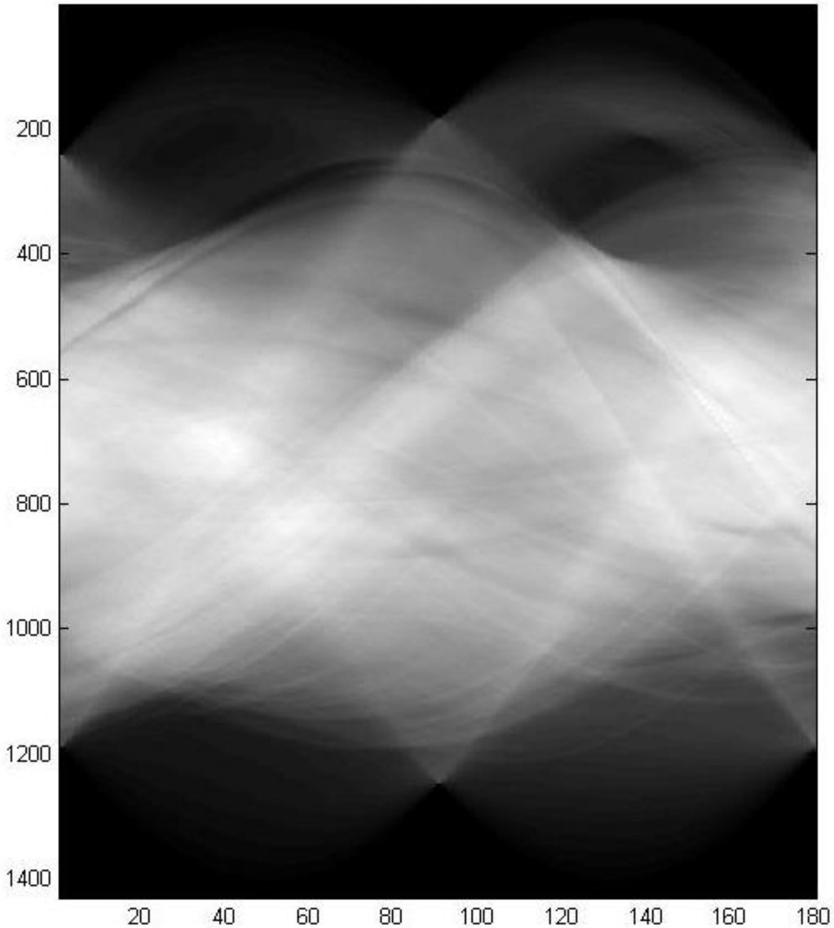


INGREDIENTS

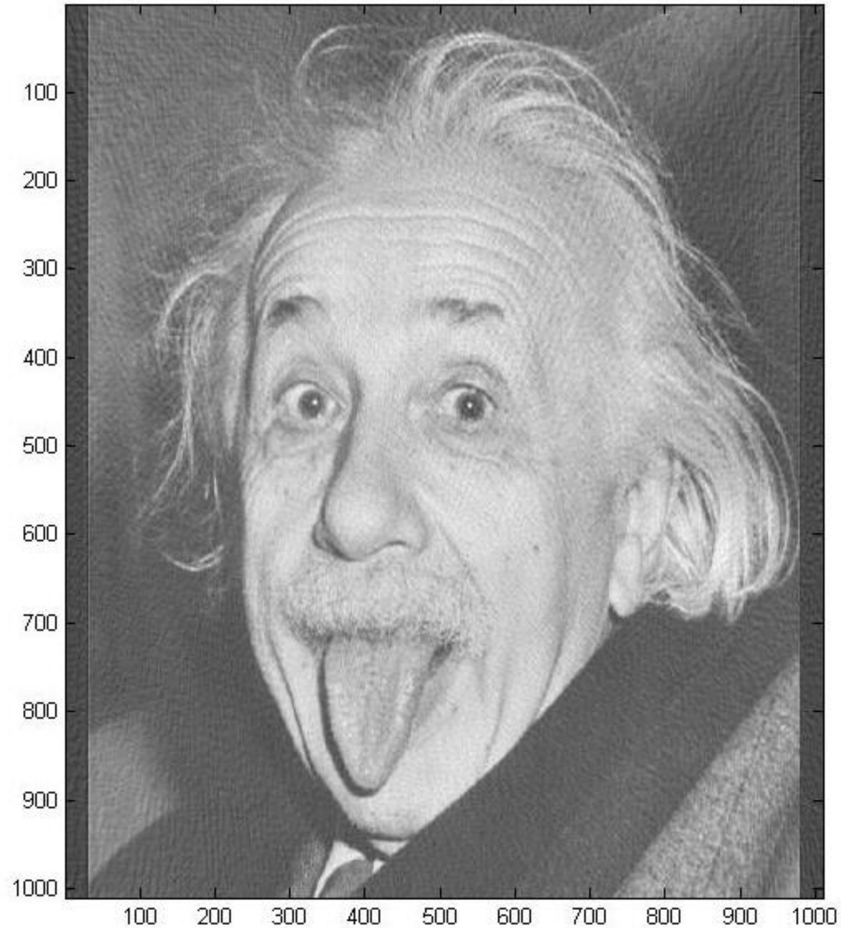
Radon Transform
Fourier Transform
Riesz operator
Back-projection
Filtering

$$f = \frac{1}{4\pi} B I^{-1} R f$$

RANGE [0,180]-STEP 1



FILTERED BACK PROJECTION



CAT-THEORY

Johann Radon (1887-1956)

1979 Nobel Prize in medicine:
Computed Axial Tomography

(Work published in 1963 to 1973)

Allan MacLeod Cormack
physicist
(1924 - 1998)

Godfrey Newbold Hounsfield
engineer
(1919-2004)

Radon model in real applications

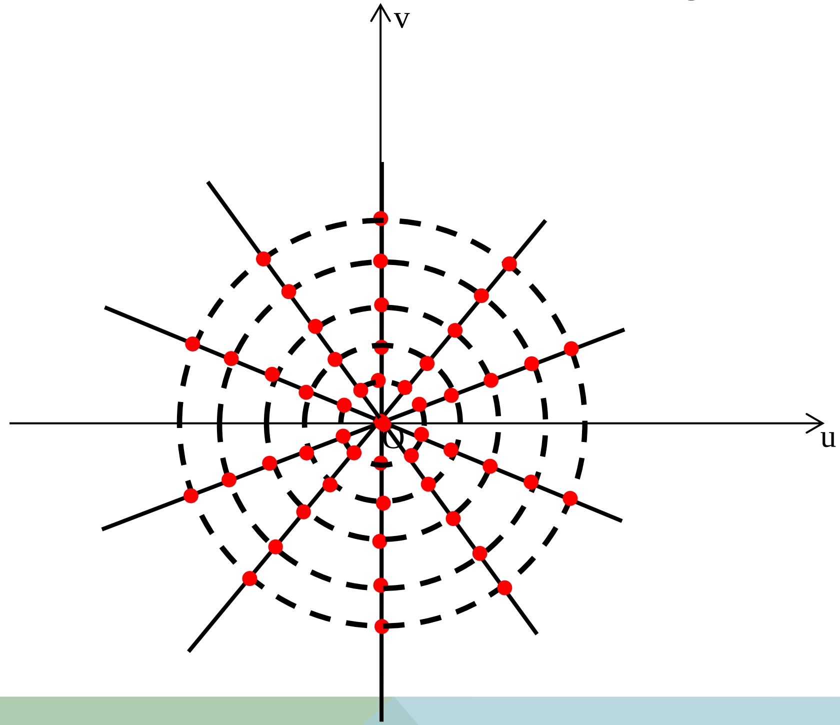
Radon model in real applications

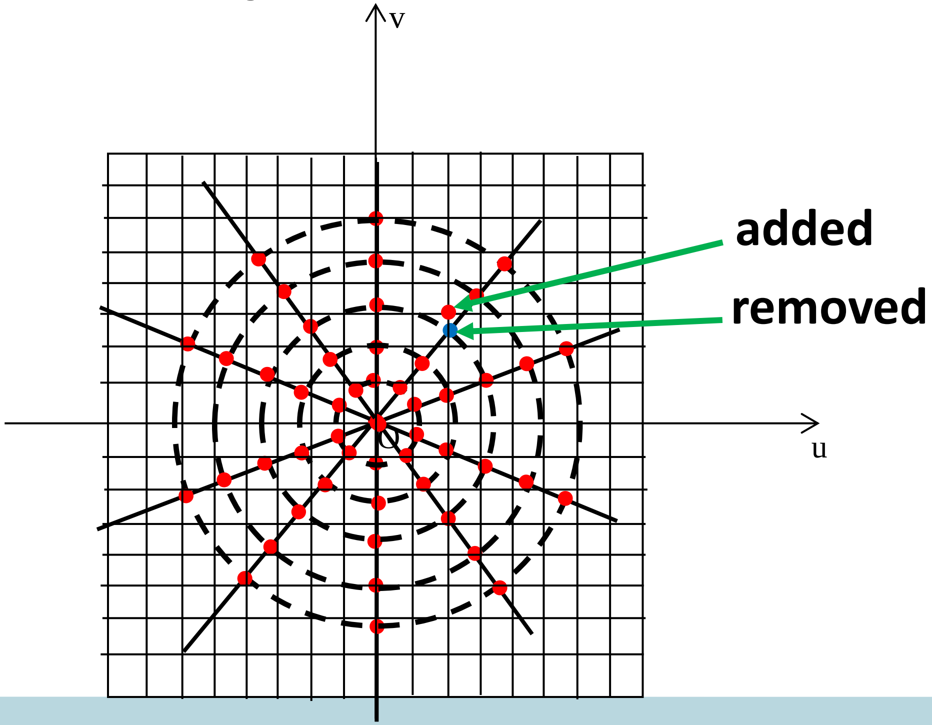
Only a finite number of projections can be collected.

Radon model in real applications

Only a finite number of projections can be collected.

Polar-cartesian interpolation is required.





Radon model in real applications

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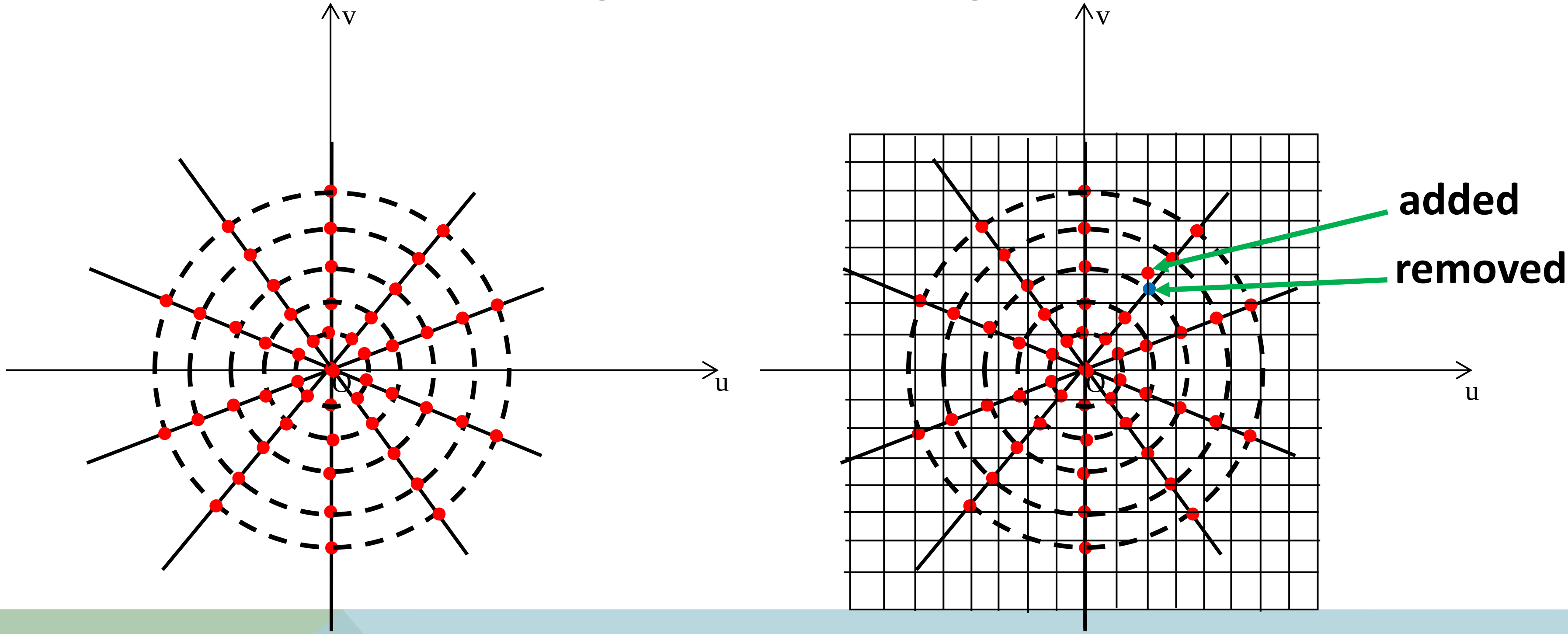
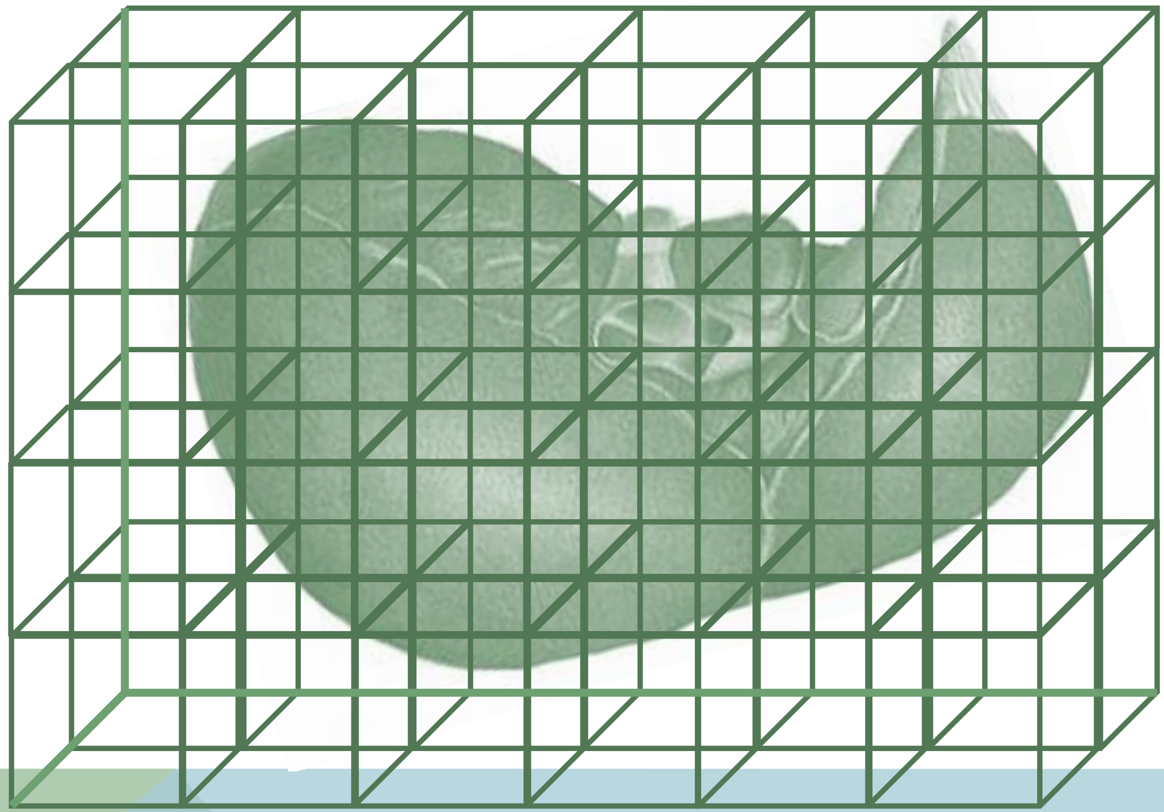


Noise
X-Ray deviation

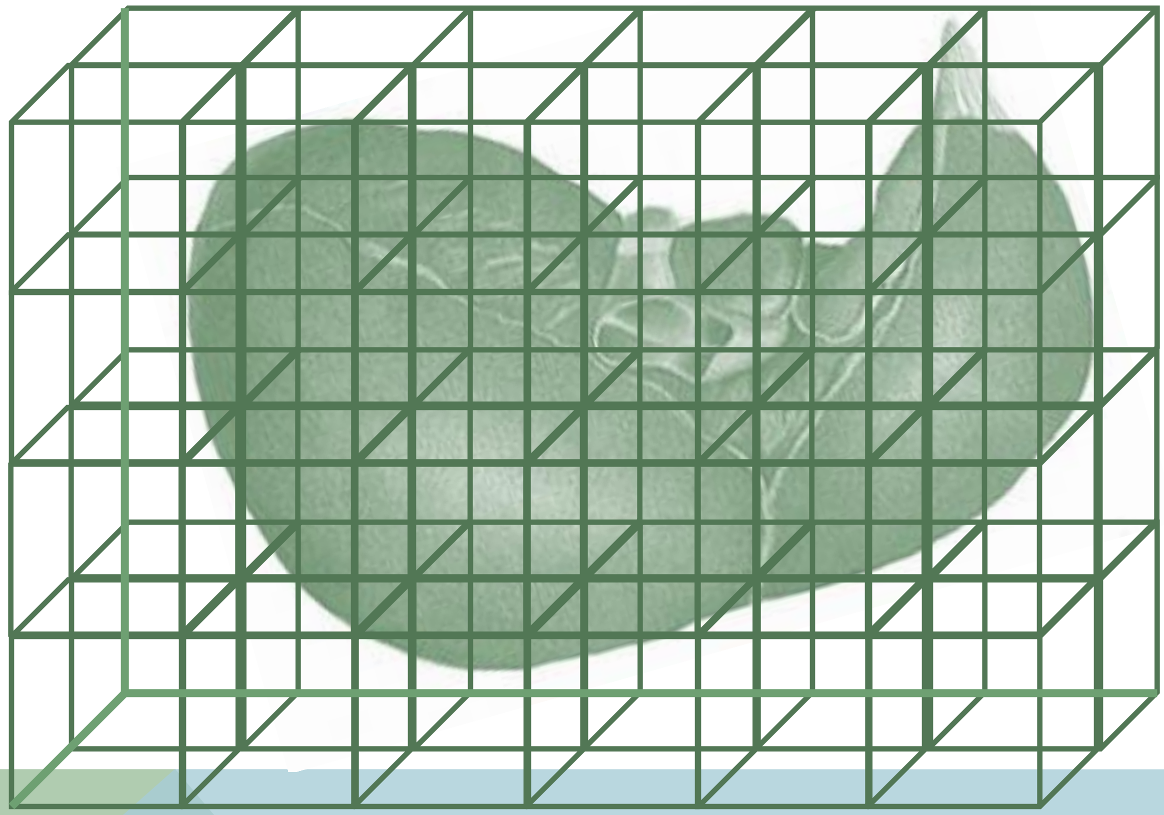
Poor quality of reconstructions

DIGITALIZATION

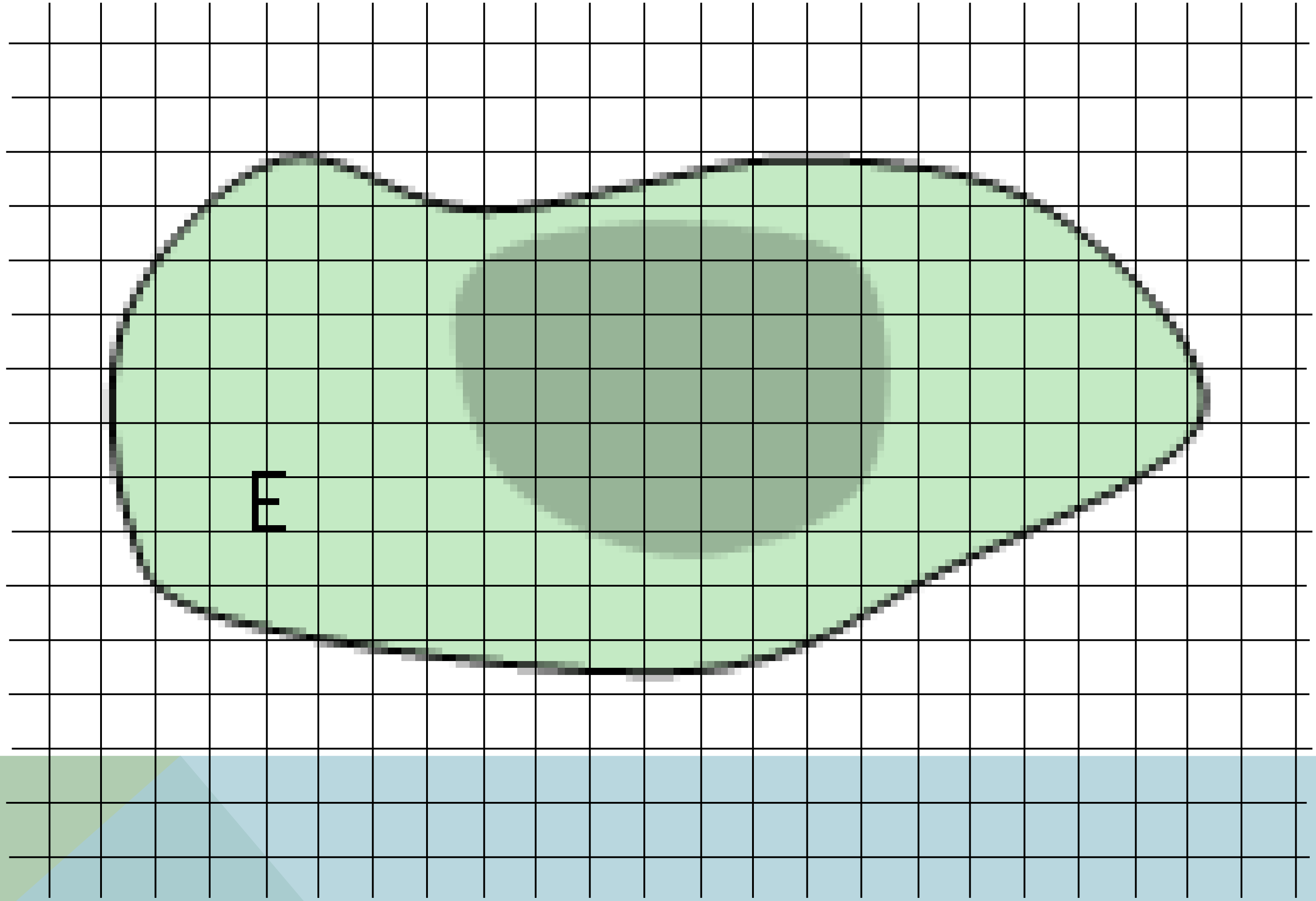
VOXELIZATION



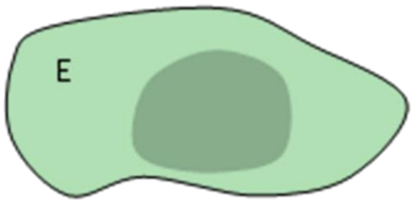
VOXELIZATION

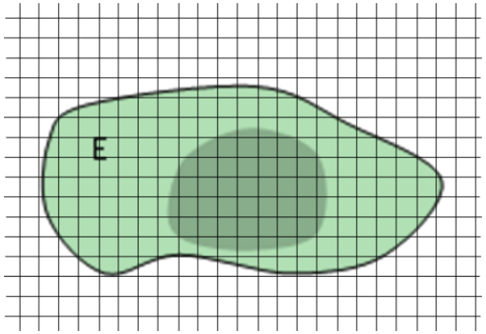


PIXELIZATION



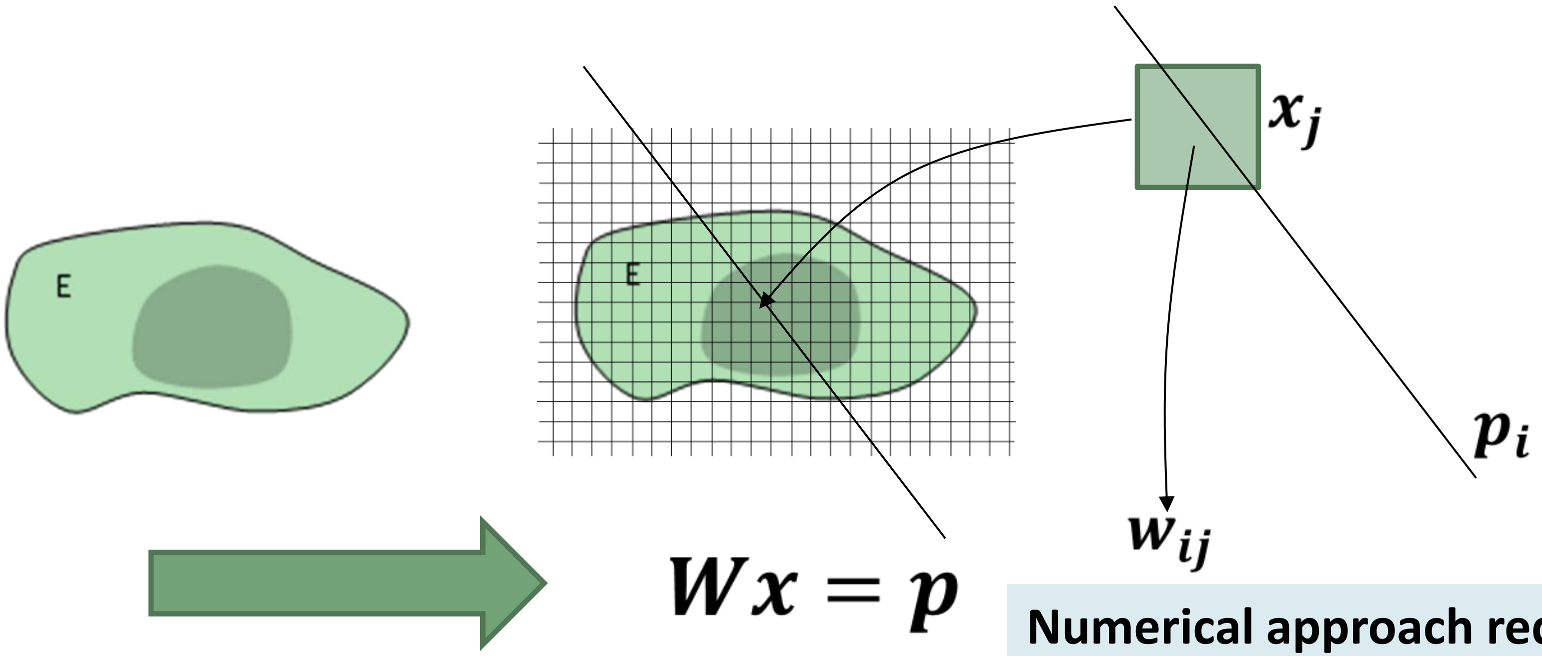
Linear System of Equations





$$\boldsymbol{W}\boldsymbol{x} = \boldsymbol{p}$$

Linear System of Equations



$$Wx = p$$

Numerical approach required

$x \in R^n$	Encoded image
$p \in R^{km}$	Projection data
$W \in R^{kmxn}$	Projection matrix
$n =$	Number of pixels
$k =$	Number of directions
$m =$	Number of detectors

Image size	Number of unknowns
10x10	100
128x128	16384
256x256	65536
512x512	262144

Example

$$X = \begin{array}{|c|c|c|}\hline ? & ? & ? \\ \hline ? & ? & ? \\ \hline \end{array}$$

Image to be reconstructed

Example

$$X = \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline \end{array}$$

Image to be reconstructed

Example

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$$

Image to be reconstructed

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

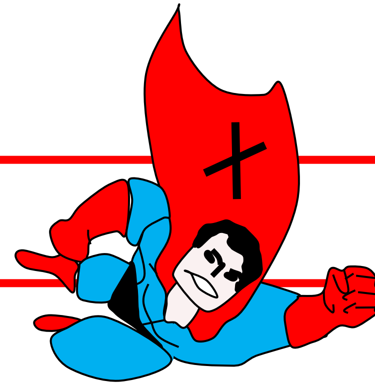
Example

$$X = \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline \end{array}$$

Let scan X along k=2 directions, say horizontal (top-bottom) and vertical (right-left)

Example

$$X = \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline \end{array}$$



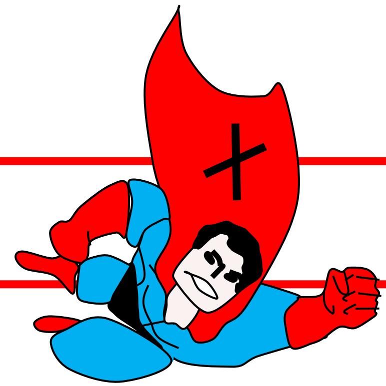
$X_1 + X_2 + X_3 = 9$

$X_4 + X_5 + X_6 = 6$

Example

$X=$

x_1	x_2	x_3
x_4	x_5	x_6



$X_1 + X_2 + X_3 = 9$

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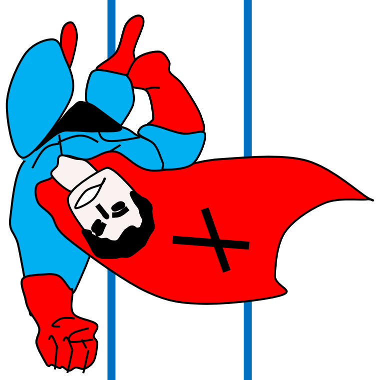
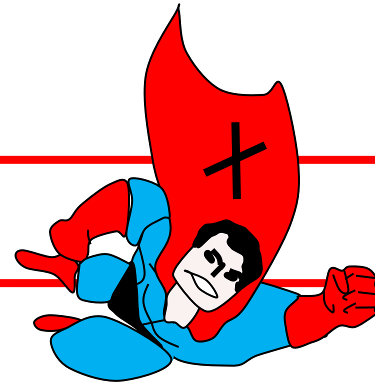


$X_1 + X_4 = 6$ $\quad$ $X_2 + X_5 = 6$ $\quad$ $X_3 + X_6 = 3$

Example

$$X = \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline \end{array}$$



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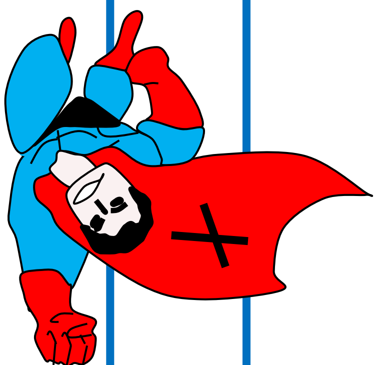


$X_1 + X_4 = 6$

$X_2 + X_5 = 6$

$X_3 + X_6 = 3$

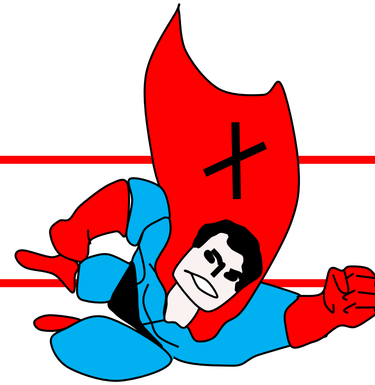
$$p = \begin{array}{|c|} \hline 9 \\ \hline 6 \\ \hline 3 \\ \hline 6 \\ \hline 6 \\ \hline \end{array}$$

projection vector

Example

$X=$

x_1	x_2	x_3
x_4	x_5	x_6



$X_1 + X_2 + X_3 = 9$

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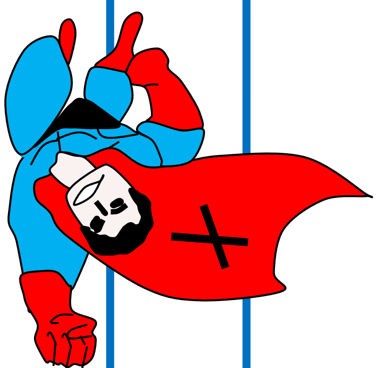


$X_1 + X_4 = 6$ $\quad X_2 + X_5 = 6$ $\quad X_3 + X_6 = 3$

$W=$

x_1	x_2	x_3	x_4	x_5	x_6
1	1	1	0	0	0
0	0	0	1	1	1
0	0	1	0	0	1
0	1	0	0	1	0
1	0	0	1	0	0

projection matrix

$$\mathbf{W} \, X = p$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \\ 6 \\ 6 \end{bmatrix}$$

$$\underset{\mathbf{W}}{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}} \underset{\boldsymbol{X}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}} = \underset{\boldsymbol{p}}{\begin{bmatrix} 9 \\ 6 \\ 3 \\ 6 \\ 6 \end{bmatrix}}$$

m=5 equations

n=6 unknown

r=rank(W)=4

$$\mathbf{W} \; X = p$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \\ 6 \\ 6 \end{bmatrix}$$

m=5 equations

n=6 unknown

r=rank(W)=4

$\Rightarrow$ ∞^2 **solutions**

$$
\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{W}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{X}
=
\underbrace{\begin{bmatrix} 9 \\ 6 \\ 3 \\ 6 \\ 6 \end{bmatrix}}_{p}
$$

$$
X = \underbrace{\begin{bmatrix} 4 & 3 & 2 \\ 2 & 3 & 1 \end{bmatrix}}_{\text{a solution image } X^*} + \underbrace{G}_{\text{any solution of } WX=0}
$$

G=Ghosts

X^*

X

$$\begin{bmatrix}4&3&2\\2&3&1\end{bmatrix} + \begin{bmatrix}4&-1&-3\\-4&1&3\end{bmatrix} = \begin{bmatrix}8&2&-1\\-2&4&4\end{bmatrix}$$

$$\begin{bmatrix}4&3&2\\2&3&1\end{bmatrix} + \begin{bmatrix}1&2&-3\\-1&-2&3\end{bmatrix} = \begin{bmatrix}5&5&-1\\1&1&4\end{bmatrix}$$

$$X^* \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X$$

$$\begin{bmatrix}4&3&2\\2&3&1\end{bmatrix} + \begin{bmatrix}4&-1&-3\\-4&1&3\end{bmatrix} = \begin{bmatrix}8&2&-1\\-2&4&4\end{bmatrix}$$

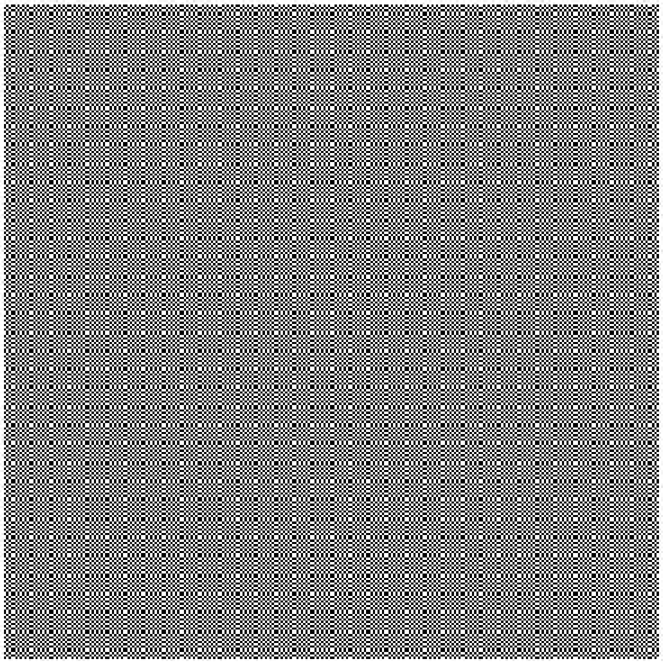
$$\begin{bmatrix}4&3&2\\2&3&1\end{bmatrix} + \begin{bmatrix}1&2&-3\\-1&-2&3\end{bmatrix} = \begin{bmatrix}5&5&-1\\1&1&4\end{bmatrix}$$

Numerical problem: compute a good approximation of a particular solution X*

Geometric Problem: investigate the space of ghosts

Ghosts corrupt the image reconstruction

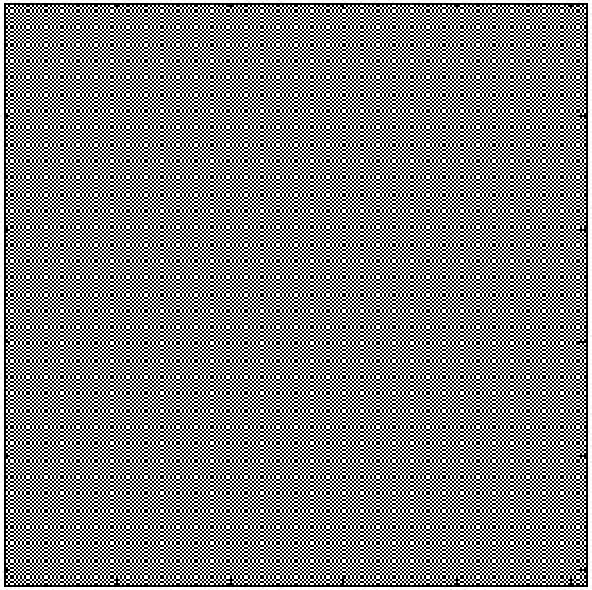
A 256x256 ghost with respect to horizontal and vertical directions



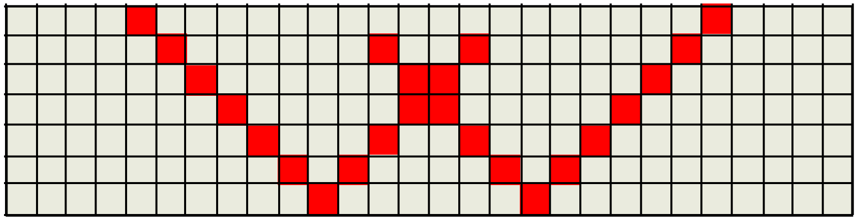
Adding ghosts provides a change in the image density

ORIGINAL

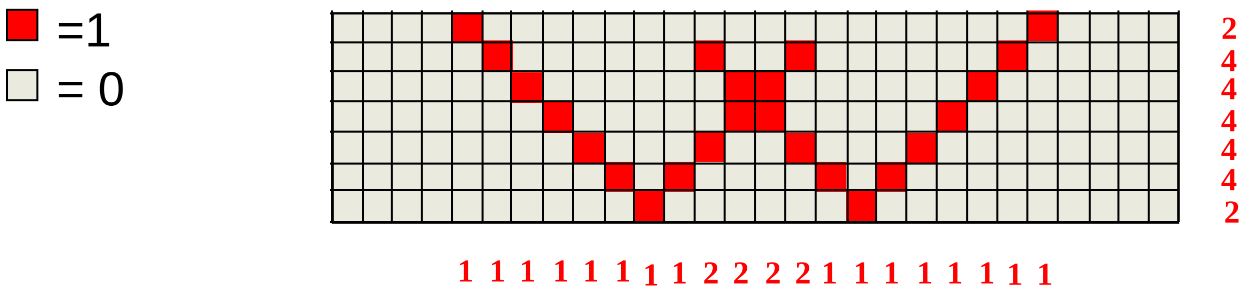
CORRUPTED



Ghosts could change completely a reconstruction

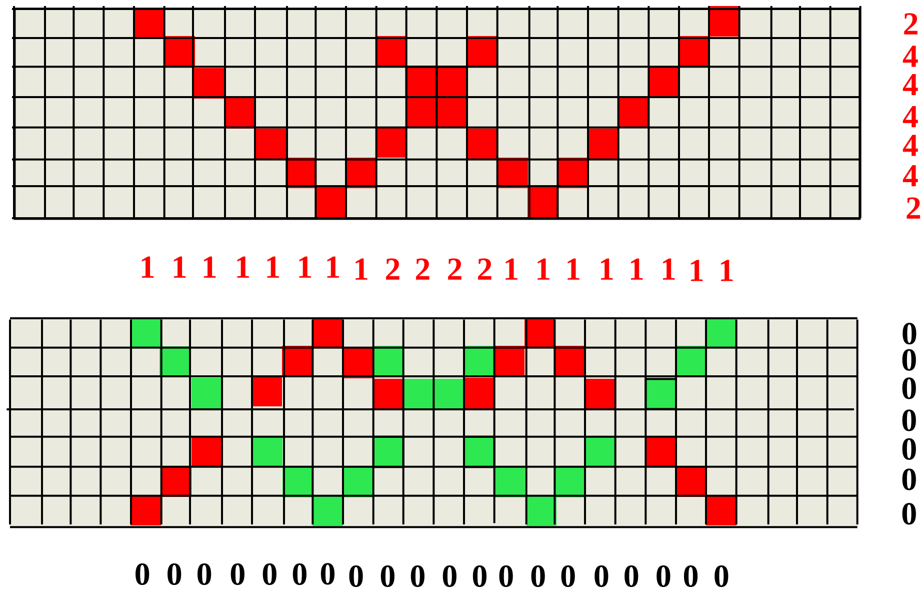


Ghosts could change completely a reconstruction



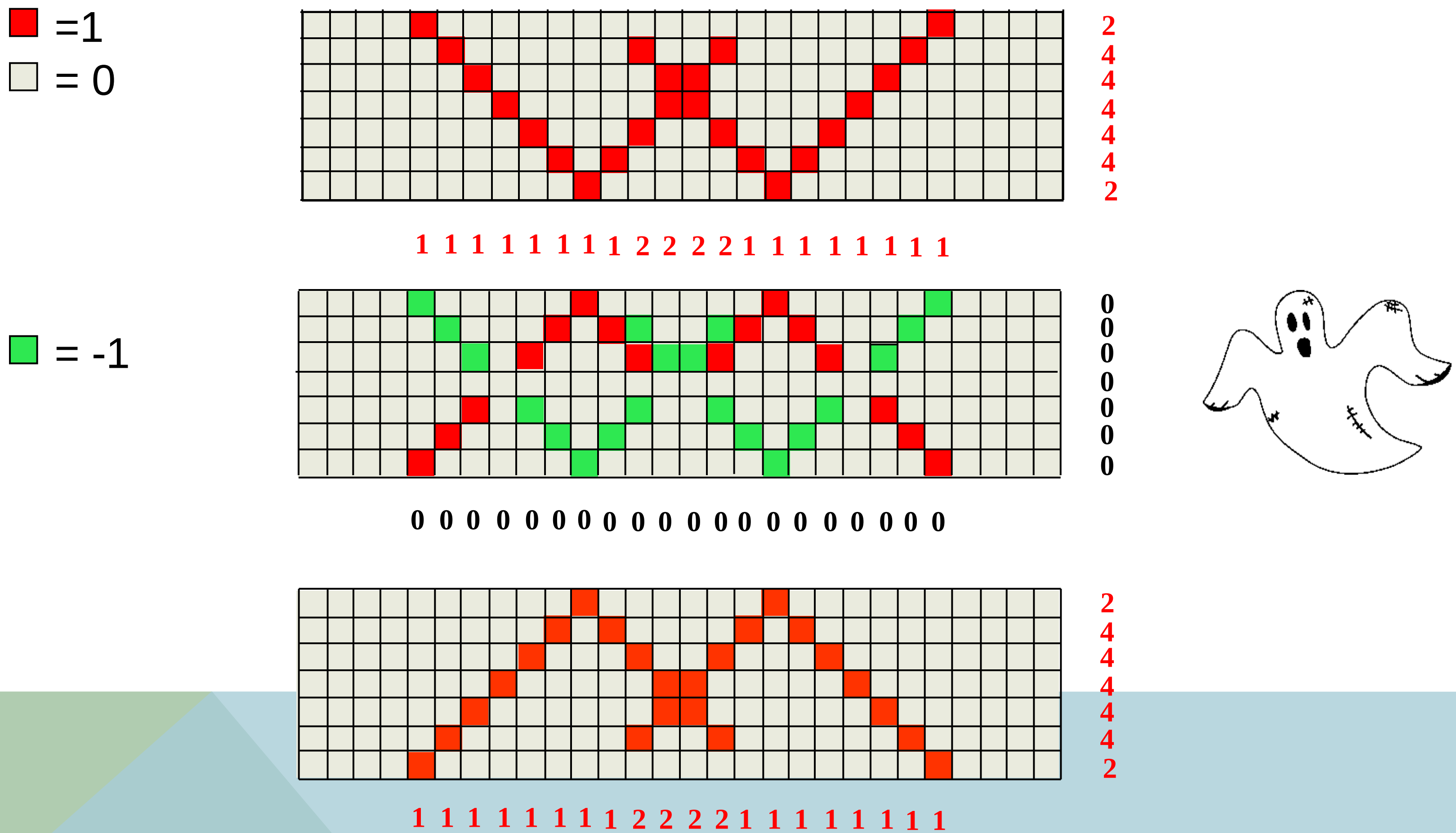
Ghosts could change completely a reconstruction

■ =1
□ = 0



■ = -1

Ghosts could change completely a reconstruction



Working with ghosts

Due to ghosts, incorporation of prior knowledge is required in the tomographic reconstruction problem.

Working with ghosts

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Tomography	Approach	Information	Space of Ghosts
Geometric Parallel X-rays	Transformations, Invariants, Geometric properties	Geometric aspects	**Geometric description** • U-polygons • Bad-configurations
Geometric Source X-rays	Measure theory	Analytic properties	**Integral description** Non-trivial zero measurable
Discrete Parallel X-rays	Polynomial factorization	• Bounding grid • Valid directions	**Algebraic description** Switching components
Discrete Source X-rays	Projective geometry Number theory	Geometric aspects	**Geometric description** P-polygons
Computerized Discrete	Algorithms based on Iterative methods	Number of grey levels, kind of noise	**Numerical description** Solutions of WX=0

Polynomial decomposition of ghosts

Assume to know a bounding lattice grid *A*. For any direction (a,b) define

$$f_{(a,b)}(x,y) = \begin{cases} x^a y^b - 1 & if\ \ a > 0, \quad b > 0 \\ x^a - y^{-b} & if\ \ a > 0, \quad b < 0 \\ x - 1 & if\ a = 1, \quad b = 0 \\ y - 1 & if\ a = 0, \quad b = 1 \end{cases}$$

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For a finite set S of directions consider the following polynomial

$$F_S(x,y) = \prod_{(a,b)\in S} f_{(a,b)}(x,y)$$

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$$F_S(x,y) = \prod_{(a,b)\in S} f_{(a,b)}(x,y)$$

For a function g: *A* $\longrightarrow$ Z define the associated polynomial

$$G_g(x,y) = \sum_{(a,b)\in A} g(a,b)x^a y^b$$

1	-2	3	4	1	5
3	-4	3	1	2	1
-3	2	1	6	5	8
1	1	2	-2	-3	4

$6x^4y^2$

Polynomial decomposition of ghosts

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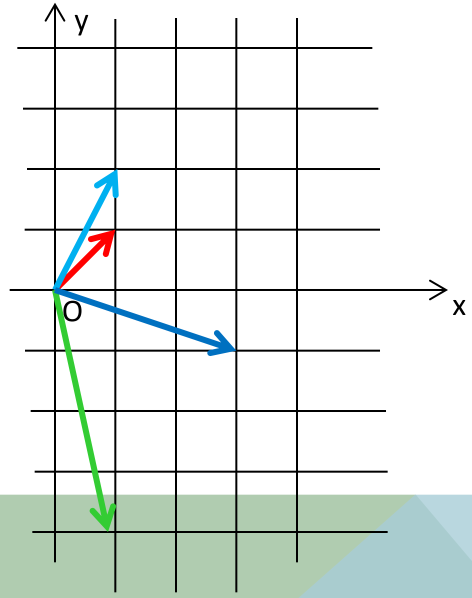
Theorem (L. Hajdu-R. Tijdeman, 2001)

The function g represents a ghost if and only if there exists H(x,y) such that

$$\boldsymbol{G_g(x,y) = F_S(x,y) H(x,y)}$$

Example

$S=\{(1,1), (1,2),(1,-4),(3,-1)\}$

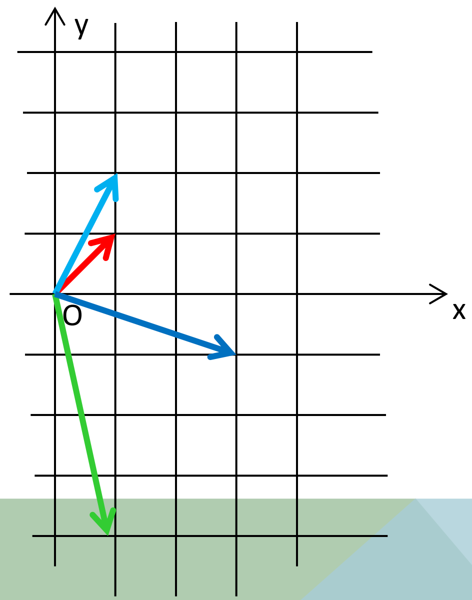


Example

S={(1,1), (1,2),(1,-4),(3,-1)}

$F_S(x,y)=$ **(xy-1) (xy²-1) (x-y⁴) (x³-y)** =

$= x^6 \cdot y^3 - x^5 \cdot y^7 - x^5 \cdot y^2 - x^5 \cdot y + x^4 \cdot y^6 + x^4 \cdot y^5 + x^4 - 2 \cdot x^3 \cdot y^4 + x^2 \cdot y^8 + x^2 \cdot y^3 + x^2 \cdot y^2 - x \cdot y^7 - x \cdot y^6 - x \cdot y + y^5$



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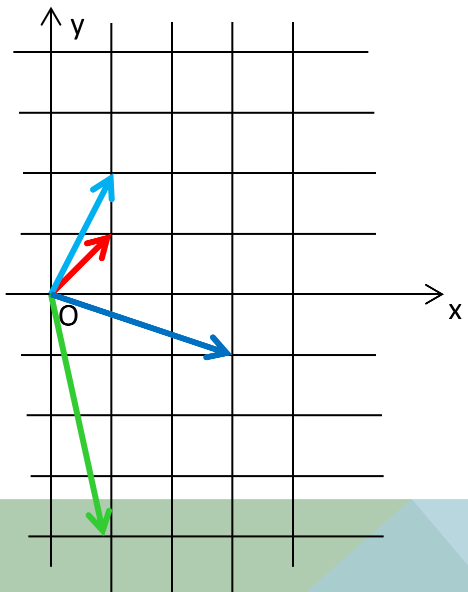
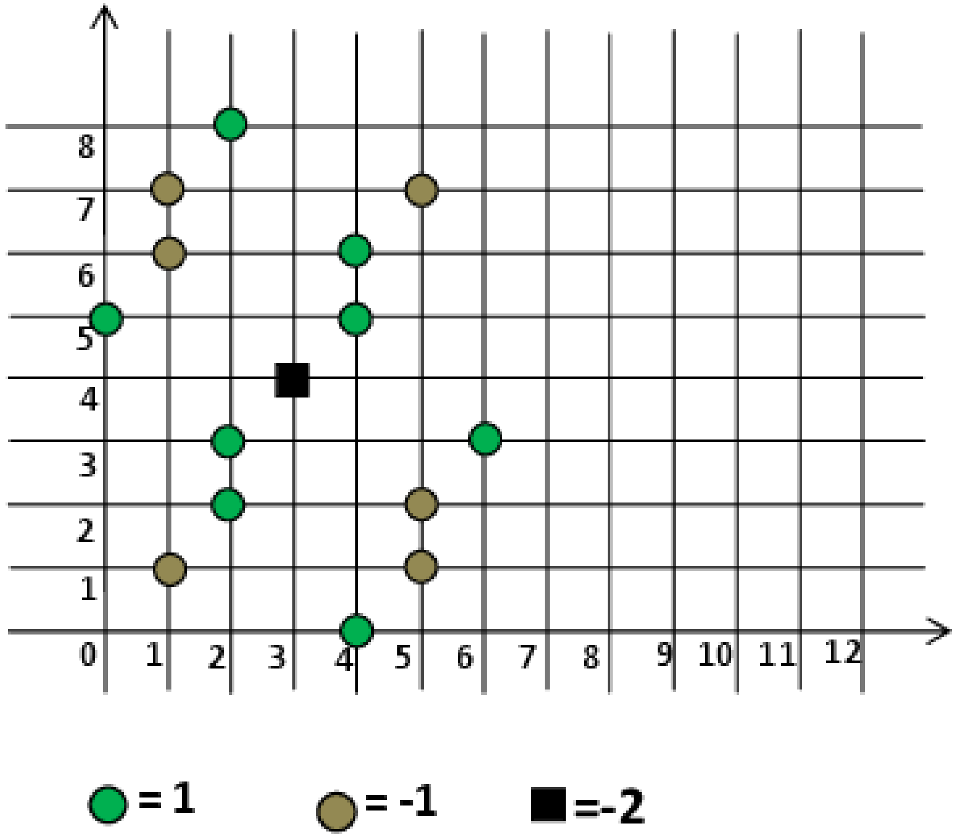
$x\cdot y^6 - x\cdot y + y^5$





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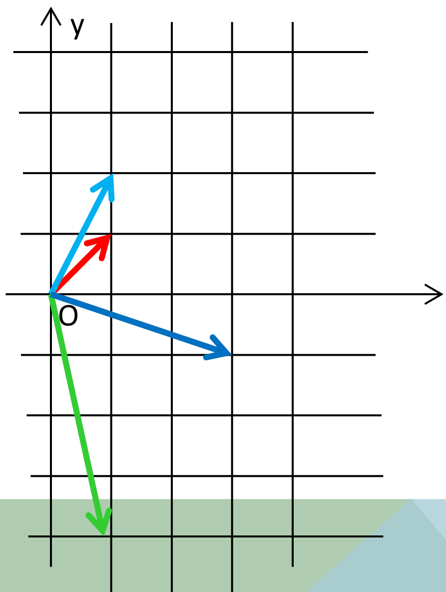

y

8	0	0	1	0	0	0	0
7	0	-1	0	0	0	-1	0
6	0	-1	0	0	1	0	0
5	1	0	0	0	1	0	0
4	0	0	0	-2	0	0	0
3	0	0	1	0	0	0	1
2	0	0	1	0	0	-1	0
1	0	-1	0	0	0	-1	0
0	0	0	0	0	1	0	0
	0	**1**	**2**	**3**	**4**	**5**	**6**

X

Example

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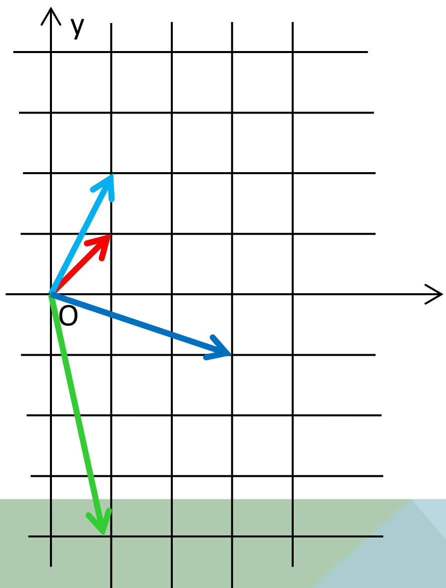

y \ X	0	1	2	3	4	5	6
8	0	0	1	0	0	0	0
7	0	-1	0	0	0	-1	0
6	0	-1	0	0	1	0	0
5	1	0	0	0	1	0	0
4	0	0	0	-2	0	0	0
3	0	0	1	0	0	0	1
2	0	0	1	0	0	-1	0
1	0	-1	0	0	0	-1	0
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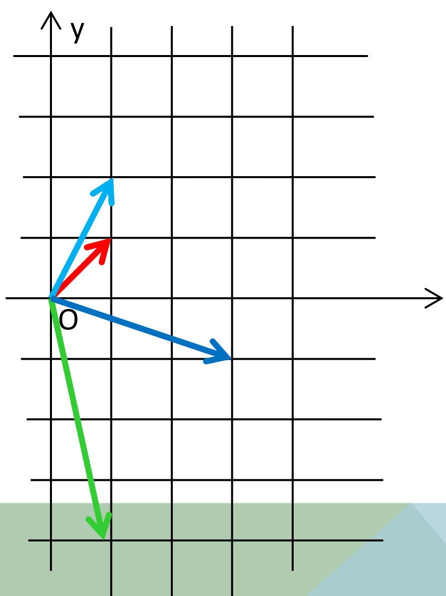

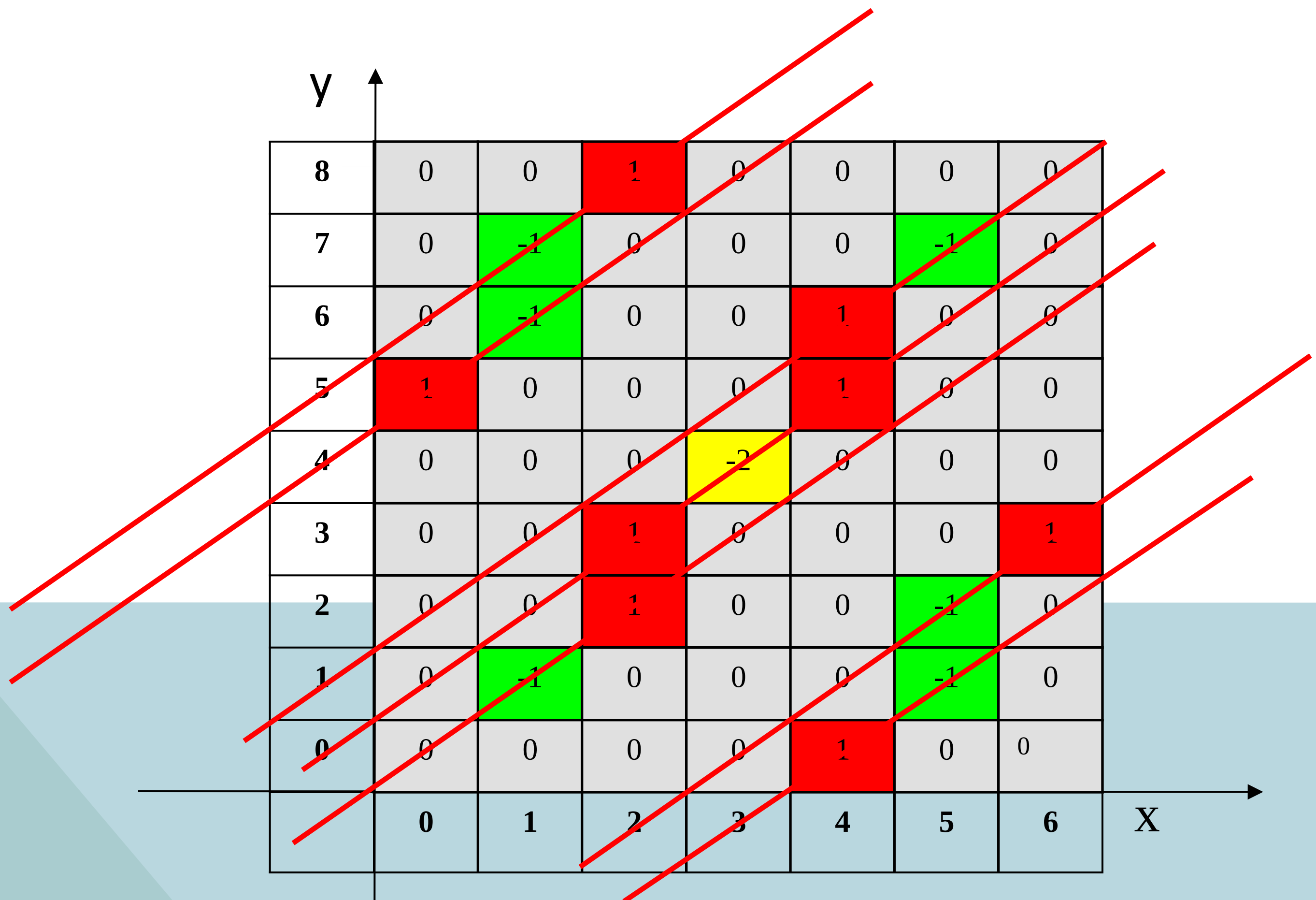

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8	0	0	1	0	0	0	0
7	0	-1	0	0	0	-1	0
6	0	-1	0	0	1	0	0
5	1	0	0	0	1	0	0
4	0	0	0	-2	0	0	0
3	0	0	1	0	0	0	1
2	0	0	1	0	0	-1	0
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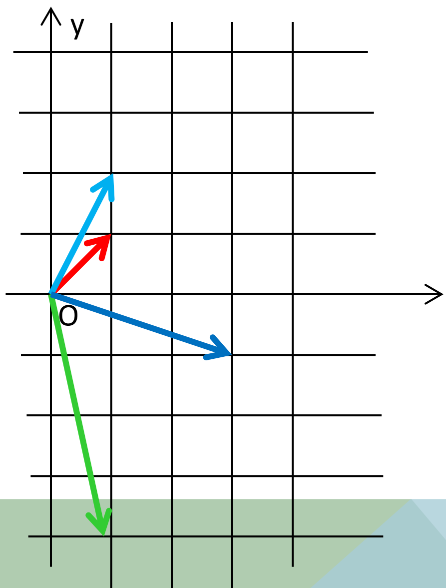

y

y \ X	0	1	2	3	4	5	6
8	0	0	1	0	0	0	0
7	0	-1	0	0	0	-1	0
6	0	-1	0	0	1	0	0
5	1	0	0	0	1	0	0
4	0	0	0	-2	0	0	0
3	0	0	1	0	0	0	1
2	0	0	1	0	0	-1	0
1	0	-1	0	0	0	-1	0
0	0	0	0	0	1	0	0
	0	**1**	**2**	**3**	**4**	**5**	**6**

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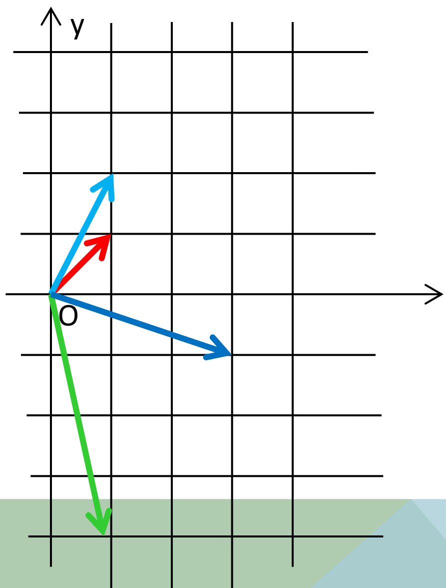
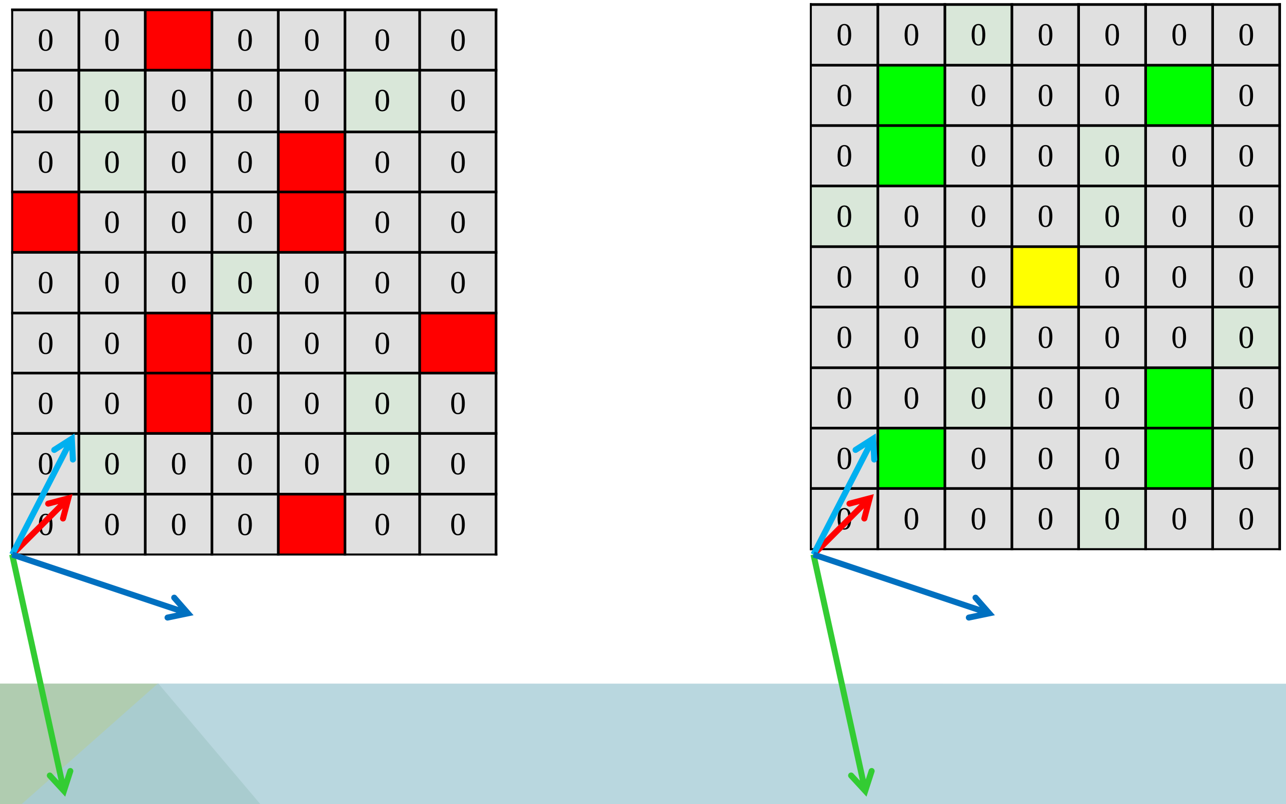


y							
8	0	0	1	0	0	0	0
7	0	-1	0	0	0	-1	0
6	0	-1	0	0	1	0	0
5	1	0	0	0	1	0	0
4	0	0	0	-2	0	0	0
3	0	0	1	0	0	0	1
2	0	0	1	0	0	-1	0
1	0	-1	0	0	0	-1	0
0	0	0	0	0	1	0	0
	0	**1**	**2**	**3**	**4**	**5**	**6**

X

Example of two sets with the same projections along the four given directions



Uniqueness of Reconstructions

Any binary set inside a given lattice grid can be uniquely reconstructed from a set $S=\{u_1, u_2, u_3, u_4=u_1+u_2\pm u_3\}$ of four suitably (precisly characterized) lattice directions.

(S. Brunetti - P. D. - C. Peri, **2013**)

Let *A* be a given lattice grid, and let S be a set of uniqueness for *A* consisting of four directions. Then any binary lattice set in *A* can be exactly reconstructed from the real valued solution X* having minimal Euclidean norm.

(P. D. - S.M. Pagani, **2018**)

Uniqueness of Reconstructions

B R A

- Take $S=\{u_1, u_2, u_3, u_4=u_1+u_2\pm u_3\}$ matching (B.D.P., **2013**)

- Compute W and p according to the directions in S  **Wx=p**

- Compute X* of minimal norm such that WX*=p (SVD, CGLS or different algorithms)

- Theorem: The binary rounding of X* solve the linear system WX=p

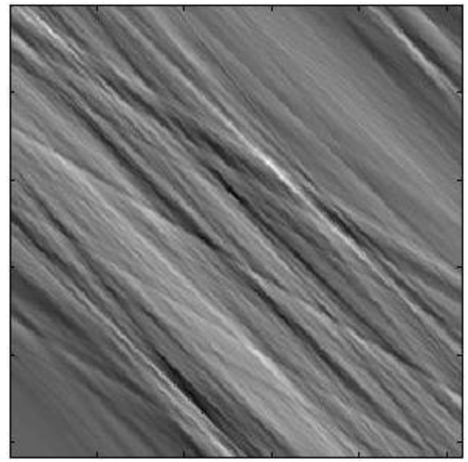
- Since S is a set of binary uniqueness, round(X*) is the desired unique reconstruction

Uniqueness of Reconstructions

X-ray width ≤ ω_S

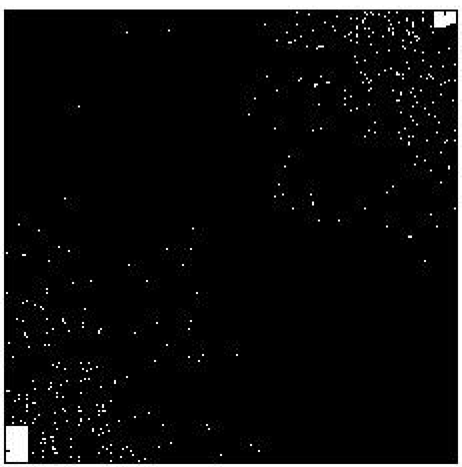
I=ORIGINAL

FBP



X*

X* - I



ROUND(X*)=I

Uniqueness of Reconstructions

X-ray width $=2\omega_S$

Iterations=154

Reconstructed=99.76%

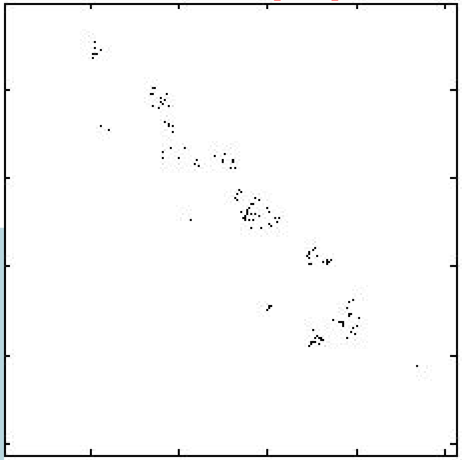
Wrong pixels=157

I=ORIGINAL

X*

ROUND(X*)

ROUND(X*)-I



Uniqueness of Reconstructions

X-ray width =3ω_S

Iterations=300

Reconstructed=98.96%

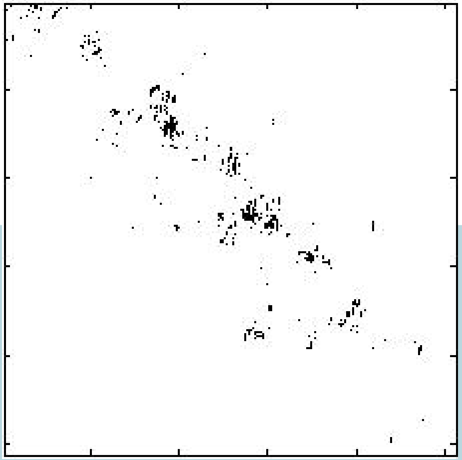
Wrong pixels=683

I=ORIGINAL

X*

ROUND(X*)

ROUND(X*)-I



Uniqueness of Reconstructions

X-ray width $=4\omega_S$

Iterations=300

Reconstructed=98%

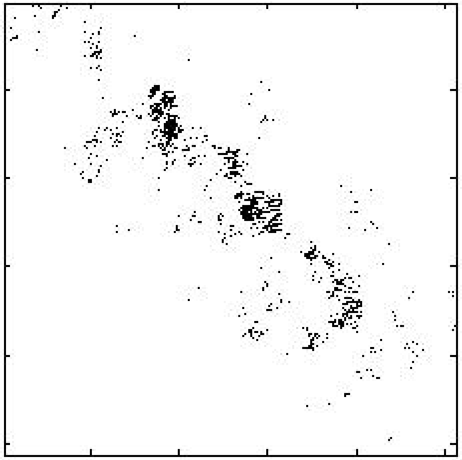
Wrong pixels=1305

I=ORIGINAL

X*

ROUND(X*)

ROUND(X*)-I




…AND THANKS FOR YOUR ATTENTION

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